Quantum Field Theory for Mathematicians
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Under Construction

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Chapter 10

Geometry in 4 dimensions: vectors, spinors and twistors

Putting space and time together, physical spacetime is four rea -dimensional. The Maxwell theory of electromagnetic fields (to be discussed in chapter 15) is formulated in terms of four-dimensional vectors and tensors, but these transform not under the group $SO(4)$ of four-dimensional rotations, but instead the Lorentz group $SO(3,1)$ of linear transformations preserving the Minkowski inner product:

$$(x, y) = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$$

The vector space $\mathbf{R}^4$ with this inner product is called "Minkowski spacetime".

Einstein's special theory of relativity was essentially the realization that not just electromagnetic fields, but the dynamics of all particles and fields should transform in the same way under the Lorentz group, replacing the classical Newtonian mechanics. In coming chapters we will see how quantum mechanics and quantum field theory need to be reformulated to have Lorentz symmetry. In this and the next chapter we'll study in detail the geometry of four dimensions, including the Minkowski geometry.

Recall that the group $SO(3)$ has a three-dimensional Lie algebra $\mathfrak{so}(3)$ of antisymmetric 3 by 3 matrices. This has basis elements l_1, l_2, l_3 given by elementary antisymmetric matrices with all entries 0 except for a 1 and a -1. One can add a row and column with index 0 and work with 4 by 4 matrices. The Lie algebra $\mathfrak{so}(3,1)$ has $\mathfrak{so}(3)$ as a Lie sub-algebra, but also three new basis elements k_1, k_2, k_3. These are symmetric and have all entries 0 except for a 1 in the index 0 column and row. One has for instance

$$l_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad k_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

One can check that the k_j transform as the components of a vector under the rotations generated by the l_j. They don't however span a Lie subalgebra. and have Lie bracket relations

$$[k_1, k_2] = -l_3, \quad [k_3, k_1] = -l_2, \quad [k_2, k_3] = -l_1$$

The k_j generate transformations of $\mathbf{R}^4$ called "boosts". For instance, exponentiating k_1 gives the linear transformation that leaves x_2, x_3 invariant and take

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \rightarrow \begin{pmatrix} \cosh & \sinh \\ \sinh & \cosh \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

The structure of $\mathfrak{so}(3,1)$ simplifies if one complexifies the Lie algebra and defines new basis elements A_j, B_j as the complex linear combinations

$$A_j = \frac{1}{2}(l_j + ik_j), \quad B_j = \frac{1}{2}(l_j - ik_j)$$

The bracket relations decouple into two identical sets for the A_j and B_j respectively, with the A_j relations

$$[A_1, A_2] = A_3, \quad [A_3, A_1] = A_2, \quad [A_2, A_3] = A_1$$

These are the Lie bracket relations for the Lie algebra $\mathfrak{sl}(2, \mathbf{C}) = \mathfrak{so}(3) \otimes \mathbf{C}$ and we have found that

$$\mathfrak{so}(3,1) \otimes \mathbf{C} = \mathfrak{sl}(2, \mathbf{C}) \oplus \mathfrak{sl}(2, \mathbf{C})$$

with the complexification breaking the Lie algebra up as the sum of two subalgebras.

In this chapter we'll study geometry in four complex dimensions, only returning to four real dimensions and Minkowski spacetime in the next chapter. We will see that there are several different ways in which going to complex dimensions clarifies and simplifies things, including

- Complex Lorentz transformations are pairs of $SL(2, \mathbf{C})$ transformations (as we saw at the Lie algebra level above).
- Allowing the time coordinate to be complex allows one todo "Wick rotation", going to imaginary time, where one recovers the usual positive definite inner product.
- Complex spacetime can be very usefully represented as 2 by 2 complex matrices, with simple behavior under complex rotations and simple relation to spinors.
- Conformal transformations of complex spacetime are simply described using the group $SL(4, \mathbf{C})$.

10.1 Complex spacetime

10.1.1 Vectors

Complex spacetime vectors in $V = \mathbf{C}^4$ with complex coordinates z_0, z_1, z_2, z_3 can be identified with the complex matrices $M(2, \mathbf{C})$ by

$$(z_0, z_1, z_2, z_3) \leftrightarrow Z = \begin{pmatrix} z_0 + z_3 & z_1 - z_2 \\ z_1 + z_2 & z_0 - z_3 \end{pmatrix} \tag{10.1}$$

If one acts on complex spacetime by the linear transformation

$$Z \to \Omega_L Z \Omega_R^{-1} \tag{10.2}$$

where Ω_L and Ω_R are complex matrices of determinant 1, such transformations preserve

$$\det Z = z_0^2 - z_1^2 + z_2^2 - z_3^2$$

so are elements of the complex orthogonal group $SO(4, \mathbf{C})$ (this would be in standard form if we changed basis by a factor of i in the 1 and 3 directions).

This gives a homomorphism mapping the product group $SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$ to $SO(4, \mathbf{C})$. It turns out that this mapping is surjective and 2 to 1 (since $(-\Omega_L, -\Omega_R)$ and (Ω_L, Ω_R) give the same transformation). We find that

$$SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R = Spin(4, \mathbf{C})$$

where $Spin(4, \mathbf{C})$ is the spin double-cover of $SO(4, \mathbf{C})$. Note that it is only in 4 dimensions that the spin group is not a simple group, but decomposes into two factors.

10.1.2 Spinors

In chapter 5 we discussed spinors in arbitary dimensions. Now we are interested in their properties in the specific case of four dimensions, which has very specific and unusual properties, due to the decomposition of $Spin(4, \mathbf{C})$ into two copies of $SL(2, \mathbf{C})$.

The group $SL(2, \mathbf{C})$ has two inequivalent spinor representations:

- The defining representation on $\mathbf{C}^2$, which we'll denote S. This representation is a holomorphic map $SL(2, \mathbf{C}) \to GL(2, \mathbf{C})$ (the inclusion map).
- The conjugate representation on $\mathbf{C}^2$ (action by conjugated matrices), which we'll denote $\overline{S}$. This representation is an anti-holomorphic map.

Note that these representations are non-unitary (the only non-trivial unitary representations of $SL(2, \mathbf{C})$ are infinite-dimensional). These representations are both unitary and unitarily equivalent to each other as representations of $SU(2) \subset SL(2, \mathbf{C})$. They are self-dual (equivalent to their dual representations). We'll later see that there an $SL(2, \mathbf{C})$ invariant nondegenerate antisymmetric bilinear form (the symplectic form) that identifies S and $\overline{S}$ with their respective duals.

Since $Spin(4, \mathbf{C})$ has two $SL(2, \mathbf{C})$ factors, it has four inequivalent spinor representations, which we'll call $S_L, \overline{S}_L, S_R, \overline{S}_R$. $S_L, \overline{S}_L$ are spinor representations of $SL(2, \mathbf{C})_L$, trivial on $SL(2, \mathbf{C})_R$, while $S_R, \overline{S}_R$ are spinor representations of $SL(2, \mathbf{C})_R$, trivial on $SL(2, \mathbf{C})_L$.

The conventional relation between vectors and spinors is to take

$$V = S_L \otimes S_R$$

defining vectors in terms of more fundamental spinor representations. Since both factors are holomorphic, this is a holomorphic representation. Equivalently, one has an identification of elements of V as complex linear maps

$$V = Hom(S_R^*, S_L)$$

with the the description 10.1 of $Z \in V$ corresponding to a particular choice of bases for S_R and S_L.

10.1.3 The Clifford algebra and antisymmetric tensors

A future version may include discussion of the complex Clifford algebra here.

10.1.4 Twistors

Twistor geometry is a 1967 proposal [19] due to Roger Penrose for a very different way of formulating four-dimensional spacetime geometry. For a detailed expository treatment of the subject, see [29]. Fundamental to twistor geometry is the twistor space $T = \mathbf{C}^4$, as well as its projective version, the space $PT = \mathbf{CP}^3$ of complex lines in T. The relation of twistor space to conventional spacetime is that complexified and conformally compactified spacetime is identified with the Grassmanian $M = G_{2,4}(\mathbf{C})$ of complex two-dimensional linear subspaces in T. A spacetime point is thus a $\mathbf{C}^2$ in $\mathbf{C}^4$ which tautologically provides the spinor degree of freedom at that point. The spinor bundle S is the tautological two-dimensional complex vector bundle over M whose fiber S_m at a point $m \in M$ is the $\mathbf{C}^2$ that defines the point.

The group $SL(4, \mathbf{C})$ acts on T and acts transitively on the spaces PT and M of its complex subspaces. Points in the Grassmanian M can be represented as elements

$$\omega = (v_1 \otimes v_2 - v_2 \otimes v_1) \in \Lambda^2(\mathbf{C}^4)$$

by taking two vectors v_1, v_2 spanning the subspace. $\Lambda^2(\mathbf{C}^4)$ is six conplex dimensional and scalar multiples of ω gives the same point in M, so ω identifies M with a subspace of $P(\Lambda^2(\mathbf{C}^4)) = \mathbf{CP}^5$. Such ω satisfy the equation

$$\omega \wedge \omega = 0 \tag{10.3}$$

which identifies (the "Klein correspondence") M with a submanifold of $\mathbf{CP}^5$ given by a non-degenerate quadratic form. Twistors are spinors in six dimensions, with the action of $SL(4,\mathbf{C})$ on $\Lambda^2(\mathbf{C}^4) = \mathbf{C}^6$ preserving the quadratic form 10.3, and giving the spin double-cover homomorphism

$$SL(4,\mathbf{C}) = Spin(6,\mathbf{C}) \to SO(6,\mathbf{C})$$

To get the tangent bundle of M, one needs not just the spinor bundle S, but also another two complex-dimensional vector bundle, the quotient bundle $S^\perp$ with fiber $S^\perp_m = \mathbf{C}^4/S_m$. Then the tangent bundle is

$$TM = Hom(S, S^\perp) = S^* \otimes S^\perp$$

with the tangent space T_mM a four complex dimensional vector space given by $Hom(S_m, S^\perp_m)$, the linear maps from S_m to $S^\perp_m$.

For a simpler analog of M, consider the space $\mathbf{C}P^1$ of complex lines in $\mathbf{C}^2$. There is also a tautological bundle over $\mathbf{C}P^1$, with fiber at each point the point itself. This bundle will be denoted L^{-1}, and it has a dual bundle denoted L. These are holomorphic line bundles and the holomorphic tangent bundle is $(L^{-1})^* \otimes L = L \otimes L \equiv L^2$. For $\mathbf{C}P^1$ one has homogeneous coordinates z_1, z_2 and can use as a coordinate $z = z_1/z_2$ away from the point where $z_2 = 0$. The conformal group $SL(2,\mathbf{C})$ acts on this coordinate by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Returning to the Grassmannian M, one can use as homogenous coordinates the 4 by 2 complex matrix

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

where Z_1, Z_2 are complex 2 by 2 matrices, giving coordinates for the complex 2-plane in $\mathbf{C}^4$ spanned by the columns. Away from planes with $\det(Z_2) = 0$, such homogeneous coordinates can be put in the form

$$\begin{pmatrix} Z \\ \mathbf{1} \end{pmatrix} \tag{10.4}$$

and the 2 by 2 complex matrix Z gives a coordinate on $M = \mathbf{G}r_{2,4}(\mathbf{C})$.

The complex conformal group $SL(4,\mathbf{C})$ acts on this coordinate by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}$$

The subgroup with $C = 0$ and $\det A = \det D = 1$ acts by

$$X \to AZD^{-1} + BD^{-1}$$

This is the exactly the action of $Spin(4,\mathbf{C})$ on complex spacetime of equation 10.2, together with a translation by BD^{-1}, giving an action of the full complex Poincaré group.

An element of twistor space T is in the complex plane corresponding to Z exactly when it is of the form

$$\begin{pmatrix} Z \\ \mathbf{1} \end{pmatrix} \pi = \begin{pmatrix} Z\pi \\ \pi \end{pmatrix}$$

for some $\pi \in \mathbf{C}^2$, since it then is a linear combination of the columns of 10.4. So, elements of T, written as

$$\begin{pmatrix} \omega \\ \pi \end{pmatrix}$$

where $\omega, \pi \in \mathbf{C}^2$ are in the plane Z when they satisfy the incidence equation

$$\omega = Z\pi \tag{10.5}$$

From the above description of $Spin(4, \mathbf{C}) = SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R \subset SL(4, \mathbf{C})$ acting on T, we see that ω is in the representation S_L, while π is in the representation S_R^*.

As a representation of $SL(2, \mathbf{C})_L \times SL(2, \mathbf{C})_R$, twistor space $T = S_L \oplus S_R^*$, which is the same thing as a Dirac spinor. But twistor space comes with additional structure, since it is an irreducible representation of a much larger group, the complex conformal group $SL(4, \mathbf{C})$.

A conventional component notation for spinors (sometimes known as the van der Waerden notation) is to write the components of spinors like ω transforming as S_L as ω^A (here $A = 1, 2$), and those transforming like S_L^* as ω_A. Indices are raised and lowered by using an $SL(2, \mathbf{C})$ invariant antisymmetric bilinear form ϵ. Transformation properties under $SL(2, \mathbf{C})_R$, are indicated in the same way, but using dotted indices. So, the components of π would be written as $\pi_{\dot{A}}$.

Since $\Lambda^2(S_L) = \Lambda^2(S_R) = \mathbf{C}$, S_L and S_R have (up to scalars) unique choices of non-degenerate antisymmetric bilinear forms, and corresponding choices of $SL(2, \mathbf{C}) \subset GL(2, \mathbf{C})$ acting on S_L and S_R. These give (again, up to scalars), a unique choice of a non-degenerate symmetric form on $S_L \otimes S_R$, such that

$$\langle Z, Z \rangle = \det Z$$

Besides the spaces PT and M of complex lines and planes in T, it is also useful to consider the correspondence space whose elements are complex lines inside a complex plane in T. This space can also be thought of as $P(S)$, the projective spinor bundle over M. There is a diagram of maps

$$\begin{array}{ccc} & P(S) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ PT & & M \end{array}$$

where ν is the projection map for the bundle $P(S)$ and μ is the identification of a complex line in S as a complex line in T. μ and ν give a correspondence between geometric objects in PT and M. One can easily see that $\mu(\nu^{-1}(m))$ is the complex projective line in PT corresponding to a point $m \in M$ (a complex two plane in T is a complex projective line in PT). In the other direction, $\nu(\mu^{-1})$ takes a point p in PT to $\alpha(p)$, a copy of $\mathbf{C}P^2$ in M, called the "α-plane" corresponding to p.

In our chosen coordinate chart, this diagram of maps is given by

$$\begin{array}{ccc} & (Z,\pi) \in P(S) & \\ \swarrow^{\mu} & & \searrow^{\nu} \\ \begin{bmatrix} Z\pi \\ \pi \end{bmatrix} \in PT & & Z \in M \end{array}$$

The incidence equation 10.5 relating PT and M implies that an α-plane is a null plane in the metric discussed above. This is because given two points Z_1, Z_2 in M corresponding to the same point in PT, their difference satisfies

$$\omega = (Z_1 - Z_2)\pi = 0$$

$Z_1 - Z_2$ is not an invertible matrix, so has determinant 0 and is a null vector.

10.2 Real forms

Physical spacetime has 4 real dimensions rather than complex dimensions. The spinor and twistor aspects of geometry in four dimensions become significantly more intricate subjects when one considers the several different possibilites for 4 real dimensional geometries complexifying to the same complex geometry considered in the previous chapter.

10.2.1 Real forms of complex representations

One normally studies Lie group representations as linear actions on a complex vector space V, but one should take into account the fact that the groups involved are real Lie groups, so one can ask about representations on real vector spaces. In some cases the groups are quaternionic and one can ask about representations on quaternionic vector spaces. The various possibilities can be studied by always working with representations on complex vector spaces and keeping track of extra structures relating these to real or quaternionic vector spaces. It turns out that there are three possibilities:

- *Real representations.* A representation on a complex vector space V is a real representation if one has a representation on a real vector space $V_{\mathbf{R}}$ such that
$$V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = V$$
This is equivalent to the existence of an anti-linear map $\sigma : V \to V$ such that $\sigma^2 = 1$. σ provides a conjugation on V and one can identify $V_{\mathbf{R}}$ as the fixed points of the σ action. In this case the representation V and the conjugate representation $\overline{V}$ are equivalent.

- *Quaternionic representations.* A representation on a complex vector space V is a quaternionic representation if one has an anti-linear map $\sigma : V \to V$ such that $\sigma^2 = -1$. In this case σ provides an action of the quaternion $\mathbf{j}$ on V. The full quaternion algebra acts on V, with the i from the action of complex numbers on V providing $\mathbf{i}$, and taking $\mathbf{k} = \mathbf{ij}$. Such representations on V are equivalent to their conjugate representation. They are sometimes called "pseudo-real" representation.

- *Complex representations.* A representation on a complex vector space V is a complex representation if it is neither real nor quaternionic. In this case V is not equivalent to its conjugate representation $\overline{V}$. Given such a V, one can form a real representation on $V \oplus \overline{V}$, taking σ to be the conjugation that interchanges V and $\overline{V}$.

An alternative point of view on this classification is that for an irreducible real representation V of a real Lie group, the argument for Schur's lemma no longer gives that $\mathrm{End}_G(V) = \mathbf{C}$, but that it can be any division algebra over $\mathbf{R}$. The classification above corresponds to the fact that the three division algebras over $\mathbf{R}$ are $\mathbf{R}, \mathbf{C}, \mathbf{H}$. For further details, see for example [20].

We will see that there are three different real forms of the complex representations on vectors, spinors and twistors of chapter 10. In all cases the vector representation is a real representation, but this will not be true for the spinors and twistors.

10.2.2 The signature $(2, 2)$ real form

One can obviously define a conjugation σ on the complex spacetime V by conjugating the matrix entries

$$\sigma \cdot \begin{pmatrix} z_0 + z_3 & z_1 - z_2 \\ z_1 + z_2 & z_0 - z_3 \end{pmatrix} = \begin{pmatrix} \overline{z}_0 + \overline{z}_3 & \overline{z}_1 - \overline{z}_2 \\ \overline{z}_1 + \overline{z}_2 & \overline{z}_0 - \overline{z}_3 \end{pmatrix}$$

by conjugating the matrix entries. Then the fixed points of σ are the real matrices

$$X = \begin{pmatrix} x_0 + x_3 & x_1 - x_2 \\ x_1 + x_2 & x_0 - x_3 \end{pmatrix}$$

The determinant of such a matrix is $x_0^2 - x_1^2 + x_2^2 - x_3^2$. Taking this as the norm-squared of an inner product, the inner product is indefinite, of signature $(2, 2)$. So we have a real spacetime $V_{2,2}$ such that

$$V_{2,2} \otimes_{\mathbf{R}} \mathbf{C} = V$$

The corresponding real form of the group $Spin(4, \mathbf{C})$ is the subgroup

$$Spin(2, 2) = SL(2, \mathbf{R})_L \times SL(2, \mathbf{R})_R$$

preserving σ. The spinor representations are also real: with the usual conjugation σ. The fixed points are the representations of $SL(2, \mathbf{R})_L$ and $SL(2, \mathbf{R})_R$ on $\mathbf{R}^2$.

Twistors are also real, with σ acting on T by the usual conjugation, with fixed points $T_{\mathbf{R}} = \mathbf{R}^4$. The real points of the compactified complex spacetime $G_{2,4}(\mathbf{C})$ are the points of the real Grassmanian $G_{2,4}(\mathbf{R})$ of real 2-planes in $\mathbf{R}^4$. The conformal group acting on this space is the real form $SL(4,\mathbf{R}) = Spin(3,3)$ of the complex spacetime conformal group $SL(4,\mathbf{C}) = Spin(6,\mathbf{C})$.

10.2.3 The signature $(4,0)$ real form: Euclidean spacetime

Euclidean spacetime is a real form V_E of complex spacetime (i.e. $V_E \otimes_{\mathbf{R}} \mathbf{C} = V$), with a positive definite inner product. The spinor representations and twistors are quaternionic, and we will begin by describing this real form in purely quaternionic terms. In these terms one can readily identify the Euclidean real forms $Sp(1) \times Sp(1) = Spin(4)$ of the complex rotation group $Spin(4,\mathbf{C})$ and $SL(2,\mathbf{H}) = Spin(5,1)$ of the complex conformal group $SL(4,\mathbf{C})$. The group $SL(2,\mathbf{H})$ is the group of quaternionic 2 by 2 matrices satisfying a single condition that one can think of as setting the determinant to one. Here one can interpret the determinant using the isomorphism with complex matrices, or, at the Lie algebra level, $\mathfrak{sl}(2,\mathbf{H})$ is the Lie algebra of 2 by 2 quaternionic matrices with purely imaginary trace.

Quaternions and four-dimensional geometry

Just as $\mathbf{C}$ is the vector space $\mathbf{R}^2$ with a basis $\{1, i\}$, and a multiplication law determined by the relation $i^2 = -1$, the quaternion algebra $\mathbf{H}$ is the vector space $\mathbf{R}^4$ with a basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and a multiplication law determined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}$$

Elements of $\mathbf{H}$ can be written as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \ q_j \in \mathbf{R}$$

The standard Euclidean norm-squared function on the vector space $\mathbf{H} = \mathbf{R}^4$ can be written in terms of quaternions as

$$|q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

where

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$

The unit norm quaternions form a group under multiplication, called $Sp(1)$, which as a manifold can be identified with the three dimensional sphere $S^3 \subset \mathbf{R}^4$. Pairs (u, v) of unit quaternions give the product group $Sp(1)_L \times Sp(1)_R$. An element (u, v) of this group acts on $q \in \mathbf{H} = \mathbf{R}^4$ by left and right quaternionic multiplication

$$q \rightarrow uqv^{-1}$$

This action preserves norms of vectors and is linear in q, so one has a homomorphism

$$\Phi : (u, v) \in Sp(1)_L \times Sp(1)_R \rightarrow \{q \rightarrow uqv^{-1}\} \in SO(4)$$

Φ is surjective, and pairs (u, v) and $(-u, -v)$ give the same element of $SO(4)$. The group $Sp(1)_L \times Sp(1)_R$ is the group $Spin(4)$, a non-trivial double cover of the group $SO(4)$. The diagonal subgroup of pairs (u, v) such that $u = v$ leaves invariant 1 and acts by an $SO(3)$ transformation on the $\mathbf{R}^3 \subset \mathbf{H}$ of imaginary quaternions. Φ restricted to this diagonal subgroup is a double cover homomorphism from the group $Spin(3) = Sp(1)$ to the group $SO(3)$.

There are two inequivalent quaternionic spinor representations of $Spin(4)$. We'll denote S_L the representation of $Spin(4)$ on $\mathbf{H}$ given by $Sp(1)_L$ acting on the left, $Sp(1)_R$ acting trivially, and S_R the representation of $Spin(4)$ on $\mathbf{H}$ given by $Sp(1)_L$ acting trivially, $Sp(1)_R$ acting on the right.

For a Euclidean spacetime version of twistor space, one can take $T = \mathbf{H}^2$, with T a quaternionic representation of the conformal group $SL(2, \mathbf{H}) = Spin(5, 1)$. A spacetime point will be a quaternionic line in $T = \mathbf{H}^2$, and spacetime M_E will be $\mathbf{H}P^1 = S^4$, the conformal compactification of the Euclidean space $\mathbf{R}^4$. The group $SL(2, \mathbf{H})$ acts transitively on $M_E = \mathbf{H}P^1 = S^4$ by conformal transformations.

Just as in the case of $\mathbf{C}P^1$, one can use as homogeneous coordinates

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where $X_1, X_2 \in \mathbf{H}$. Away from $X_2 = 0$, these can be put in the form

$$\begin{pmatrix} X \\ \mathbf{1} \end{pmatrix}$$

with $X \in \mathbf{H}$. The conformal group $SL(2, \mathbf{H})$ acts by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot X = (AX + B)(CX + D)^{-1}$$

where now $A, B, C, D \in \mathbf{H}$. The Euclidean group in four dimensions will be the subgroup of elements of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

such that A and D are independent unit quaternions, thus in the group $Sp(1)$, and B is an arbitrary quaternion. The Euclidean group acts by

$$X \rightarrow AXD^{-1} + BD^{-1}$$

with the spin double cover of the rotational subgroup now $Spin(4) = Sp(1) \times Sp(1)$.

Relating quaternionic and complex

While we have seen that the translations, rotations and conformal transformations of four dimensional Euclidean geometry can be efficiently understood purely in terms of quaternions, it is often desirable to instead work with complex quantities, together with an antilinear σ satisfying $\sigma^2 = -1$ on quaternionic representations and $\sigma^2 = 1$ on real representations (note that one gets real representations when working with quaternions since the tensor product of two quaternionic representations is real).

To identify $\mathbf{H}$ with $\mathbf{C}^2$, there are various choices to be made:

- One can identify $\mathbf{C}$ as the subalgebra of $\mathbf{H}$ spanned by $1, u$, where u is any element satisfying $u^2 = -1$. There is an S^2 of possibilities (any unit length linear combination of the purely imaginary quaternions).
- Choosing a $v \in \mathbf{H}$ such that $v^2 = -1$ and $uv = -vu$ gives a $\mathbf{C}$-basis of $\mathbf{H}$, so an identification with $\mathbf{C}^2$.

The conventional choices made are: $u = \mathbf{i}$ (giving a consistent meaning for the symbol "i") and $v = \mathbf{j}$. Then an arbitrary quaternion can be written as

$$q = z_1 + \mathbf{j}z_2$$

or as a vector

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Here one is also making the choice that, as a complex vector space, the subalgebra of $\mathbf{H}$ of complex numbers acts on $\mathbf{H}$ on the right. As a complex spinor representation of $Sp(1)$, the group acts on the left, with a commuting action of $\mathbf{H}$ on the right. This will be a quaternionic representation, with a standard choice of σ right multiplication by $\mathbf{j}$. Since

$$(z_1 + \mathbf{j}z_2)\mathbf{j} = \mathbf{j}\overline{z}_1 + \mathbf{j}^2\overline{z}_2 = -\overline{z}_2 + \mathbf{j}\overline{z}_1$$

σ acts by

$$\sigma : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_2 \end{pmatrix}$$

On the Euclidean version of twistor space, one has $T = \mathbf{C}^4$, with quaternionic structure map σ given by

$$\sigma : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \rightarrow \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_2 \\ -\overline{z}_3 \\ \overline{z}_4 \end{pmatrix} \tag{10.6}$$

The group $SL(2, \mathbf{H})$ acts on this quaternionic representation, which is just the complex form of the action on $\mathbf{H}^2$.

Given this identification of $\mathbf{H}$ with $\mathbf{C}^2$, one can use the left action of $\mathbf{H}$ on this $\mathbf{C}^2$ to get an isomorphism of algebras between $\mathbf{H}$ and the subalgebra of $M(2, \mathbf{C})$ of matrices of the form

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \tag{10.7}$$

for $\alpha, \beta \in \mathbf{C}$. As an algebra , $M(2, \mathbf{C})$ has two inequivalent real forms: $M(2, \mathbf{R})$ and $\mathbf{H}$ are non-ismorphic algebras satisfying

$$M(2, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} = M(2, \mathbf{C})$$

The usual conjugation of complex matrices has fixed points $M(2, \mathbf{R})$. An inequivalent conjugation σ on $M(2, \mathbf{C})$ corresponding to the real form $\mathbf{H}$ is given by

$$\sigma \cdot \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} \overline{\delta} & -\overline{\beta} \\ -\overline{\gamma} & \overline{\alpha} \end{pmatrix}$$

This satisfies $\sigma^2 = 1$ and is clearly antilinear, with fixed points of the form 10.7.

More explicitly, the identification 10.7 takes

$$1 \leftrightarrow \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Physicists often like to use instead the Pauli matrices, taking

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} \leftrightarrow -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{j} \leftrightarrow -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{k} \leftrightarrow -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

The correspondence between $\mathbf{H}$ and 2 by 2 complex matrices is then given by

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \leftrightarrow \begin{pmatrix} q_0 - iq_3 & -(q_2 + iq_1) \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix}$$

In general we'll avoid choosing between the mathematicians and physicists by avoiding an explicit choice of one of the two identifications above.

Since

$$\det \begin{pmatrix} q_0 - iq_3 & -(q_2 + iq_1) \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

we see that the length-squared function on quaternions corresponds to the determinant function on 2 by 2 complex matrices. Taking $q \in Sp(1)$, so of length one, the corresponding complex matrix is in $SU(2)$.

Still to do? Understand vectors as a tensor product $\mathbf{h} = \mathbf{H} \otimes_{\mathbf{H}} \mathbf{H}$, explicitly how the spinor representation as a right action works. The usual conjugation on quaternions in terms of matrices?

Projective twistor space and Euclidean twistors

The projective twistor space PT is fibered over S^4 by complex projective lines

$$\begin{array}{ccc} \mathbf{C}P^1 & \longrightarrow & PT = \mathbf{C}P^3 \\ & & \big\downarrow \pi \\ & & S^4 = \mathbf{H}P^1 \end{array} \tag{10.8}$$

The projection map π is just the map that takes a complex line in T identified with $\mathbf{H}^2$ to the corresponding quaternionic line it generates (multiplying elements by arbitrary quaternions). In this case the conjugation map σ of 10.6 has no fixed points on PT, but does fix the complex projective line fibers and thus the points in $S^4 \subset M$. The action of σ on a fiber takes a point on the sphere to the opposite point, so has no fixed points.

In the Euclidean case, the projective twistor space has another interpretation, as the bundle of orientation preserving orthogonal complex structures on S^4. A complex structure on a real vector space V is a linear map J such that $J^2 = -1$, providing a way to give V the structure of a complex vector space (multiplication by i is multiplication by J). J is orthogonal if it preserves an inner product on V. While on $\mathbf{R}^2$ there is just one orientation-preserving orthogonal complex structure, on $\mathbf{R}^4$ the possibilities can be parametrized by a sphere S^2. The fiber $S^2 = \mathbf{C}P^1$ of 10.8 above a point on S^4 can be interpreted as the space of orientation preserving orthogonal complex structures on the four real dimensional tangent space to S^4 at that point.

One way of exhibiting these complex structures on $\mathbf{R}^4$ is to identify $\mathbf{R}^4 = \mathbf{H}$ and then note that, for any real numbers x_1, x_2, x_3 such that $x_1^2 + x_2^2 + x_3^2 = 1$, one gets an orthogonal complex structure on $\mathbf{R}^4$ by taking

$$J = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

Another way to see this is to note that the rotation group $SO(4)$ acts on orthogonal complex structures, with a $U(2)$ subgroup preserving the complex structure, so the space of these is $SO(4)/U(2)$, which can be identified with S^2.

More explicitly, in our choice of coordinates, the projection map is

$$\pi : \begin{bmatrix} s \\ s^\perp = Zs \end{bmatrix} \to Z = \begin{pmatrix} x_0 - ix_3 & -ix_1 - x_2 \\ -ix_1 + x_2 & x_0 + ix_3 \end{pmatrix}$$

For any choice of s in the fiber above Z, $s^\perp$ associates to the four real coordinates specifying Z an element of $\mathbf{C}^2$. For instance, if $s = \big(1, 0\big)$, the identification of $\mathbf{R}^4$ with $\mathbf{C}^2$ is

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} x_0 - ix_3 \\ -ix_1 + x_2 \end{pmatrix}$$

The complex structure on $\mathbf{R}^4$ one gets is not changed if s gets multiplied by a complex scalar, so it just depends on the point $[s]$ in the $\mathbf{C}P^1$ fiber.

For another point of view on this, one can see that for each point $p \in PT$, the corresponding α-plane $\nu(\mu^{-1}(p))$ in M intersects its conjugate $\sigma(\nu(\mu^{-1}(p)))$ in exactly one real point, $\pi(p) \in M^4$. The corresponding line in PT is the line determined by the two points p and $\sigma(p)$. At the same time, this α-plane provides an identification of the tangent space to M^4 at $\pi(p)$ with a complex two plane, the α-plane itself. The $\mathbf{C}P^1$ of α -planes corresponding to a point in S^4 are the different possible ways of identifying the tangent space at that point with a complex vector space.

The correspondence space $P(S)$ (here the complex lines in the quaternionic line specifying a point in $M^4 = S^4$) is just PT itself, and the twistor correspondence between PT and S^4 is just the projection π. In the Euclidean case the action of the real form $SL(2, \mathbf{H})$ is transitive on PT.

10.2.4 The $(3,1)$ real form: Minkowski spacetime

The Maxwell equations describing electromagnetism (see section **??**) are invariant under the group $SO(3,1)$ acting on spacetime, taken to be the Minkowski spacetime $\mathbf{R}^{3,1}$, the four dimensional space $\mathbf{R}^4$ with an indefinite inner product given by

$$(x, y) \equiv x \cdot y = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$$

(here x_j, y_j are coordinates on $\mathbf{R}^4$, with $j = 0$ the time coordinate). Einstein's discovery of special relativity was based on the observation that for consistency one should describe not just electromagnetism but also mechanics in a formalism based on taking spacetime to be $\mathbf{R}^{3,1}$, with physical laws invariant under $SO(3,1)$.

Vectors $v \in \mathbf{R}^{3,1}$ such that $|v|^2 = v \cdot v > 0$ are called "space-like", those with $|v|^2 < 0$ "time-like" and those with $|v|^2 = 0$ are said to lie on the "light cone". Suppressing one space dimension, the picture to keep in mind of Minkowski spacetime looks like this:

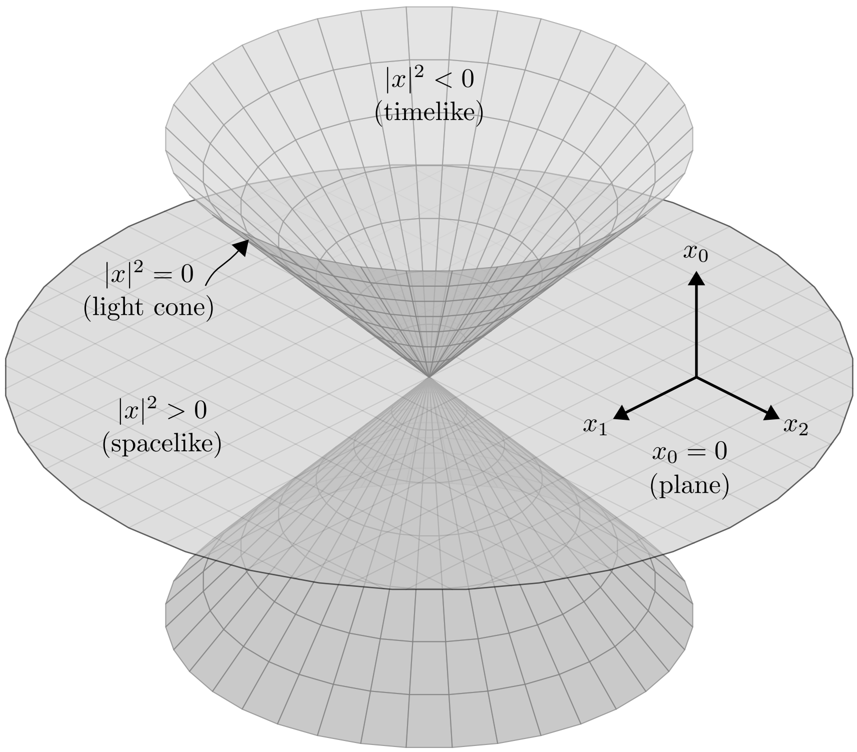

Figure 10.1: Light cone structure of Minkowski spacetime.

Like $\mathbf{R}^{2,2}$ and $\mathbf{H}$, $\mathbf{R}^{3,1}$ is a real form of $M(2,\mathbf{C})$. The conjugation σ is given by

$$\sigma \cdot Z = -Z^\dagger$$

with fixed points the skew-Hermitian matrices, of the form

$$X = (-i)\begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

which have determinant

$$\det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

The subgroup of $Spin(4,\mathbf{C}) = SL(2,\mathbf{C})_L \times SL(2,\mathbf{C})_R$ that commutes with the action of σ and thus preserves skew-Hermiticity is the group $SL(2,\mathbf{C})$, with $\Omega \in SL(2,\mathbf{C})$ acting by

$$X \to \Omega X \Omega^\dagger$$

where $\Omega^\dagger$ is the adjoint (conjugate transpose) of Ω. Recall that $SL(2,C)$ has two kinds of spinor representations: S (action by Ω) and the conjugate representation $\overline{S}$ (action by $\overline{\Omega}$). Vectors in Minkowski spacetime thus transform under the Lorentz group $SL(2,\mathbf{C})$ as the tensor product $S \otimes \overline{S}$.

Explain that $SL(2, \mathbf{C})$ is double cover of component of $SO(3, 1)$ preserving time orientation.

Spinors have some quite different properties in Minkowski spacetime than in the signature $(2, 2)$ and Euclidean cases. These $SL(2, \mathbf{C})$ representations are not real or quaternionic, but complex, so there is no antilinear $\sigma : S \to S$ or $\sigma : \overline{S} \to \overline{S}$ commuting with $SL(2, \mathbf{C})$. What there is instead is an antilinear map σ from S to S^*, which is a map of $SL(2, \mathbf{C})$ representations

$$\sigma : S \to \overline{S}^*$$

This takes a representation matrix Ω to $(\Omega^\dagger)^{-1}$ and satisfies $\sigma^2 = 1$. σ gives a real structure on the $SL(2, \mathbf{C})$ representation $S \oplus \overline{S}^*$ which interchanges the terms in the direct sum. This real $SL(2, \mathbf{C})$ representation is known to physicists as the Majorana representation. On σ fixed points it is an $SL(2, \mathbf{C})$ representation on a 4-real dimensional vector space, equivalent to considering $SL(2, \mathbf{C}$ as a real Lie group, and $\mathbf{C}^2$ as a real vector space (*check this*).

The twistor geometry in the Minkowski signature case also has different properties. As in the case of spinors, twistor space T is a complex representation of $SL(4, \mathbf{C})$, and one needs to consider not just T with an antilinear map σ, but T and T^* with an antilinear map between them. Such a map σ will give an identification of T and $\overline{T}^*$, and so a non-degenerate Hermitian form Φ on T. This picks out a unitary subgroup of $SL(4, \mathbf{C})$ which turns out to have signature $(2, 2)$. So, in this case, the real form of the complex conformal group is the conformal group $SU(2, 2) = Spin(4, 2)$.

The conformal compactification of Minkowski space is a real submanifold of M, denoted here by $M^{3,1}$. It is acted upon transitively by the conformal group $Spin(4, 2) = SU(2, 2)$. This conformal group action on $M^{3,1}$ is most naturally understood using twistor space, as the action on complex planes in T coming from the action of the real form $SU(2, 2) \subset SL(4, \mathbf{C})$ on T.

$SU(2, 2)$ is the subgroup of $SL(4, \mathbf{C})$ preserving a real Hermitian form Φ of signature $(2, 2)$ on $T = \mathbf{C}^4$. In our coordinates for T, a standard choice for Φ is given by

$$\Phi\left(\begin{pmatrix}\omega \\ \pi\end{pmatrix}, \begin{pmatrix}\omega' \\ \pi'\end{pmatrix}\right) = \begin{pmatrix}\overline{\omega} & \overline{\pi}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\omega' \\ \pi'\end{pmatrix} = \omega^\dagger\pi' + \pi^\dagger\omega' \tag{10.9}$$

Minkowski space is given by complex planes on which $\Phi = 0$, so

$$\Phi\left(\begin{pmatrix}X\pi \\ \pi\end{pmatrix}, \begin{pmatrix}X\pi \\ \pi\end{pmatrix}\right) = \pi^\dagger(X + X^\dagger)\pi = 0$$

(recall that X are skew-Hermitian matrices).

One can identify compactified Minkowski space $M^{3,1}$ as a manifold with the Lie group $U(2)$ which is diffeomorphic to $(S^3 \times S^1)/\mathbf{Z}_2$. The identification of the tangent space with anti-Hermitian matrices reflects the usual identification of the tangent space of $U(2)$ at the identity with the Lie algebra of anti-Hermitian matrices.

$SL(4,\mathbf{C})$ matrices are in $SU(2,2)$ when they satisfy

$$\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Poincaré subgroup P of $SU(2,2)$ is given by elements of $SU(2,2)$ of the form

$$\begin{pmatrix} A & B \\ 0 & (A^\dagger)^{-1} \end{pmatrix}$$

where $A \in SL(2,\mathbf{C})$ and $A^\dagger B = -B^\dagger A$. These act on Minkowski space by

$$X \to (AX + B)A^\dagger$$

$BA^\dagger$ is anti-Hermitian and gives arbitrary translations on Minkowski space. The Lorentz subroup is $Spin(3,1) = SL(2,\mathbf{C})$ acts by

$$X \to AXA^\dagger$$

Here $SL(2,\mathbf{C})$ is acting by the standard representation on S, and by the conjugate-dual representation on $\overline{S}^*$.

The $SU(2,2)$ action on M has six orbits: $M_{++}, M_{--}, M_{+0}, M_{-0}, M_{00}$, where the subscript indicates the signature of Φ restricted to planes corresponding to points in the orbit. The last of these is a closed orbit $M^{3,1}$, compactified Minkowski space. Acting on projective twistor space PT, there are three orbits: PT_+, PT_-, PT_0, where the subscript indicates the sign of Φ restricted to the line in T corresponding to a point in the orbit. The first two are open orbits with six real dimensions, the last a closed orbit with five real dimensions. The points in compactified Minkowski space $M_{00} = M^{3,1}$ correspond to projective lines in PT that lie in the five dimensional space PT_0. Points in M_{++} and M_{--} correspond to projective lines in PT_+ or PT_- respectively.

10.3 For further reading

Among the places one can find more details of the material in this chapter, see [29] and [16].