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THEORY OF ELECTROMAGNETIC FIELD FROM A HIGH-ALTITUDE SHOT

James Paul Wesley

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James Paul Wesley

Lawrence Radiation Laboratory, University of California

Livermore, California

ABSTRACT

The present investigation concerns the theoretical derivation of the altitude effect electromagnetic field produced by a nuclear bomb explosion. The electric polarization wave, which is produced by gamma rays from the bomb knocking Compton electrons radially outward from the bomb, progresses through the air where the density varies with altitude. The variation of the air density with altitude causes the electric polarization wave to be nonspherically symmetrical and to be the source of a large electromagnetic field - the altitude effect. There is a brief discussion of the physical mechanisms that establish the polarization wave. And there is an approximate derivation of the two physical parameters of interest: the electric polarization source strength, A, and the reciprocal effective mean free path of gamma rays in air, β.

The theoretically derived electric polarization wave source is substituted into Maxwell's equations. By introducing a scalar, ψ, Maxwell's equations reduce to a single inhomogeneous scalar wave equation. Applying Green's theorem the scalar ψ and consequently the electric and magnetic field components are expressed in the form of triple integrals over a spheroidal volume which grows in time. These integrals are presented in forms suitable for numerical integration; however, no numerical results are presented. After the passage of the polarization wave the static charge distribution produces a static electric field. This static electric field is expressed in the form of double integrals which involve the complete elliptic integrals.

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James Paul Wesley
Lawrence Radiation Laboratory, University of California
Livermore, California

1. INTRODUCTION

This paper is incomplete in the sense that the numerical integrations yielding the electric and magnetic field components as functions of time are not included. But since it may be some time, if ever, before the difficult and arduous numerical work is completed, and since the work accomplished thus far is of value, it was decided to present the material without the final numerical results. Whenever the numerical integrations are completed a supplementary report will be issued.

This study is primarily concerned with obtaining an order-of-magnitude estimate of the electromagnetic fields to be expected from a high-altitude nuclear bomb explosion; and it is not intended to be complete analysis. Gamma rays which are emitted by a nuclear bomb explosion displace Compton electrons radially outward from the bomb. This process produces an electric polarization wave which travels radially outward from the bomb with the velocity of light. If no nonspherical features are present, only a small radial electric field corresponding to the emf of the process will be observed. However, if the electric polarization wave becomes nonspherical, very large electromagnetic fields result. The spherical symmetry is disturbed if the earth interrupts the polarization wave. The large fields that result are attributed to the boundary effect which is treated in another paper (Wesley, ref. 8). The spherical symmetry is also destroyed if the polarization wave travels through the air where the density changes with altitude. This altitude effect is treated in the present paper. The altitude effect in the presence of a bounding earth may also be treated; but has been reserved for future investigation. Here we are involved with the pure altitude effect involving no boundaries, as is appropriate for a high-altitude shot.

2. SOURCE

The detailed analysis of all of the processes from the initial emission of gamma rays from the bomb to the final production of an electromagnetic field by the Compton electrons is extremely difficult. The present analysis must, therefore, rely in part upon some average parameters, approximations, intuition, and guess work.

2a. Initial Gamma Rays

The gamma rays initially emitted by the bomb are emitted according to some distribution in time, energy, and direction. Here we assume for mathematical convenience that all of the gamma rays have the same energy which is chosen to be 2 Mev. Because of the large times involved we are justified in assuming that the gamma rays are emitted as a delta function burst in time which then proceed radially outward with the velocity of light. c,

$$\delta\,(ct-r)\ , \tag{2.1}$$

where t is the time and r is the radial distance from the center of the bomb,

$$r^2 = \rho^2 + |z - h|^2\ , \tag{2.2}$$

where ρ is the cylindrical radial distance, z the height of the observer, and h the height of the bomb.

We also assume that the bomb is spherically symmetrical and that the gamma rays are emitted spherically uniformly. The only assumption likely to produce serious difficulty is the assumption of monoenergetic gamma rays.

2b. Attenuation of Gamma Rays

Only the component of the gamma rays moving radially outward from the bomb is assumed to be effective in producing a charge separation, and therefore an electromagnetic signal. Tangential components of the gamma rays being equivalent in all tangential directions will produce ionization but no net charge separation. Thus, the attenuation of gamma rays in space and time of interest here will not be the distribution functions so carefully worked out for gamma ray dosage.

In particular, assuming no capture or attenuation, the number of gamma rays, n, passing through a unit area on the surface of a sphere of radius r about the bomb is,

$$n = N/4\pi r^2 , \tag{2.3}$$

where N is the total number of gamma rays emitted by the bomb. The number absorbed per unit volume of air is proportional to the density of the air, the effective mass cross section, and to the total number of gamma rays passing through a unit area, n,

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n) = - \beta n , \tag{2.4}$$

where β is the effective mass cross section times the density.

Since the density of the air, μ, varies with height, z, we assume the functional form

$$\mu = \mu_o' \, e^{-kz} , \tag{2.5}$$

where μ_o' is the density of air at sea level and k is the reciprocal scale height. Then β becomes

$$\beta = \beta_o' \, e^{-kz} . \tag{2.6}$$

Substituting Eq. (2.6) into (2.4), letting

$$z = h + r \cos \theta \, , \tag{2.7}$$

and integrating with respect to r, we obtain

$$\ln (r^2 n) = \beta_o' \, r \, \frac{e^{-kz}}{k(z-h)} + C(\theta) \; . \tag{2.8}$$

The function of θ, $C(\theta)$, may be determined by requiring that for $\theta = \pi/2$

$$\ln (r^2 n) = \beta_o' r \, e^{-kh} + \ln \frac{N}{4\pi} \qquad \text{where } \theta = \pi/2. \tag{2.9}$$

Thus,

$$C(\theta) = - \beta_o' \frac{e^{-kh}}{k \cos \theta} \ln \frac{N}{4\pi} \tag{2.10}$$

$$= - \beta_o' r \frac{e^{-kh}}{k \, (z-h)} \ln \frac{N}{4\pi} \; ;$$

and Eq. (2.8) yields,

$$n = \frac{N}{4\pi} \frac{1}{r^2} \exp \left[- \beta_o' r \, \frac{e^{-kh} - e^{-kz}}{k \, (z-h)} \right] . \tag{2.11}$$

If we include the variation with time as specified by the delta function, Eq. (2.1), we obtain the appropriate approximate distribution function for the gamma rays in space and time.

$$n = \frac{N}{4\pi} \, \delta \, (ct-r) \, \frac{1}{r^2} \exp \left[- \beta_o' r \, \frac{e^{-kh} - e^{-kz}}{k \, (z-h)} \right] . \tag{2.12}$$

This result may be easily generalized to include the presence of a layer which is either opaque to electromagnetic fields, such as an ionized layer of atmosphere, or which absorbs the gamma rays, such as the surface of the earth, by including the step function factor,

$$s(z) = \begin{cases} 1 & \text{for } z \geqslant 0, \\ 0 & \text{for } z < 0. \end{cases} \tag{2.13}$$

2c. Initial Electrons

The number of Compton electrons generated as a function of time and space is assumed to be proportional to the number of gamma rays absorbed, or the right side of Eq. (2.4) where Eq. (2.12) gives the value of n.

In detail a gamma ray of energy $E = h\nu$ gives up its momentum $h\nu/c$ to a number of electrons and atoms. Assuming the momentum transferred to atoms is negligible we have

$$\frac{h\nu}{c} = m_o \sum_{i=1}^{n'} \frac{v_i \cos \alpha_i}{\left(1 - \frac{v_i^2}{c^2}\right)^{\frac{1}{2}}} \tag{2.14}$$

where m_o is the electrons rest mass, v_i is the speed of the ith electron and α_i is the angle the velocity vector of the ith electron makes with respect to the original radial direction of the gamma ray.

The gamma ray also shares its original energy among the same electrons and atoms. If we assume that the amount of energy transferred to the atoms is also negligible, we have from conservation of energy,

$$h\nu = m_o c^2 \sum_{i=1}^{n'} \left[\left(1 - \frac{v_i^2}{c^2}\right)^{-\frac{1}{2}} - 1\right] \tag{2.15}$$

It is true that the initial points at which the n' electrons are generated by the multiple scattering of the gamma ray may be very widely separated in space. Due to the spherical geometry there will, thus, be a tendency for a larger number of less energetic Compton electrons to be produced further from the bomb. It is also true that the gamma ray which has been scattered with a tangential component will no longer have as large a radial component of velocity; we may, thus, expect straggling in time. Despite the actual presence of these phenomena to alter the space and time distribution of the initial Compton electrons, we will assume that we have a time and space distribution as given by the right side of Eq. (2.4) and Eq. (2.12). Later we will introduce a term which permits some straggling in time irrespective of the cause of the straggling. The geometrical effect is assumed to be zero insofar as the final electromagnetic field is concerned, since less energetic electrons produce less of an effect.

2d. Attenuation of Electrons

Each of the n' Compton electrons moves through space, losing momentum and energy by colliding with electrons and by producing bremsstrahlung, until it finally becomes attached to an atom and produces a negative ion.

Since we are interested in this process as an electromagnetic source we are interested in only two questions: (1) What is the net charge separation? and (2) What is the time to produce such a net charge separation? Ignoring the detailed processes we assume each Compton electron is displaced from its original atom (left positive) by a distance equal to the range of the electron in air.

The number of electrons surviving after passing through the distance x is given approximately by,

$$n_e = n_o \, e^{-x/\lambda} , \tag{2.16}$$

where λ is the mean effective range which is a function of the air density,

$$\lambda = \lambda_o' \, e^{+kz} , \tag{2.17}$$

and is a function of the energy of the Compton electron.

2e. Polarization Wave

The prescribed polarization produced by the Compton electrons and cascaded electrons is the electric dipole moment per unit volume or is the sum

$$\vec{P} = - e \sum_{i=1}^{n_o} \vec{r}_i , \tag{2.18}$$

where n_o is the total number of electrons per unit volume which is proportional to the number of gamma rays absorbed per unit volume as specified by Eq. (2.12) and the right side of Eq. (2.4), e is the electron charge, and $\vec{r}_i$ is the displacement of the electron from the positive ion left behind. The displacement $\vec{r}_i$ starts abruptly in time when a gamma ray strikes the electron and is then assumed to change uniformly with time.

In particular, we assume that the summation in Eq. (2.18) can be represented by a survival distribution of electrons as given by Eq. (2.16)

$$\vec{P} = - e \vec{e}_r n_o \lambda (1 - e^{-x/\lambda}) , \tag{2.19}$$

where $\vec{e}_r$ is the unit vector in the radial direction. Only the radial component of $\vec{P}$ survives, since all other directions of motion for the electrons produce no net charge separation. Since we are assuming that the polarization remains indefinitely once the electrons have become attached, we have used the function $(1 - e^{-x/\lambda})$ in Eq. (2.19) instead of $e^{-x/\lambda}$. The distance x appearing in Eq. (2.19) is assumed to be radial and to be varying uniformly with time,

$$x = vt_o , \tag{2.20}$$

where v is a mean effective velocity in the x direction. The time t_o is measured from the moment the Compton electron is produced; thus,

$$t_o = t - r/c , \tag{2.21}$$

where r is measured from the center of the bomb.

Making the substitutions (2.20) and (2.21) in Eq. (2.19), we have the polarization wave,

$$\vec{P} = - \vec{e}_r n_o e \lambda S (ct-r) , \tag{2.22}$$

where

$$S(ct-r) \equiv \begin{cases} 0 & \text{for } ct-r < 0 , \\ 1 - e^{-\sigma(ct-r)} & \text{for } ct-r \geqslant 0 , \end{cases} \tag{2.23}$$

where we have introduced the single parameter

$$\sigma = v/c\lambda \ . \tag{2.24}$$

Substituting in the variation of λ with height, Eq. (2.17), we obtain

$$\sigma = \sigma_o' e^{-kz} \ ,$$

where

$$\sigma_o' = v/c\lambda_o' \ . \tag{2.25}$$

When $\lambda \to 0$ or $\sigma \to \infty$, S(ct-r) goes over to the step function S(ct-r) as it should (see Fig. 1).

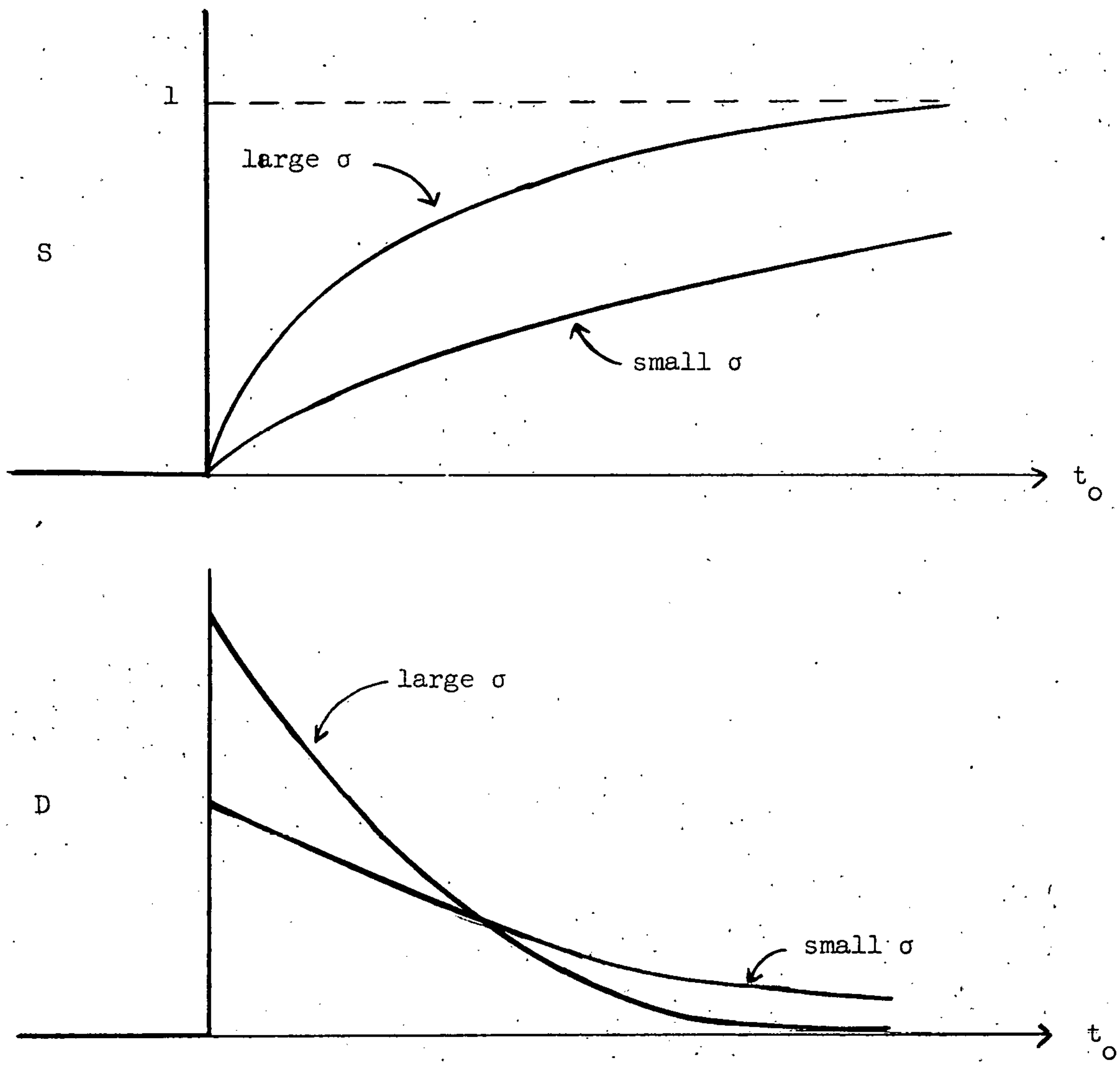

Fig. 1

Pictorial representation of the functions $S(t_o)$ and $D(t_o)$.

Since n_o is proportional to the number of gamma rays absorbed, we have

$$n_o = A'\beta n \ , \tag{2.26}$$

where n is given by Eq. (2.12), neglecting the delta function. Substituting Eqs. (2.26) and (2.12) without the delta function into Eq. (2.22), we obtain

$$P = - e_r A \ S(ct-r) \ \frac{e^{-\gamma r}}{r^2} \ , \tag{2.27}$$

where S(ct-r) is defined by Eq. (2.23), γ is a function of z defined by

$$\gamma = \beta_o' \ \frac{e^{-kh} - e^{-kz}}{k \ (z-h)} \ , \tag{2.28}$$

and A is not a function of z,

$$A = \frac{N}{4\pi} \ eA'\lambda_o'\beta_o' \ . \tag{2.29}$$

We may simplify Eq. (2.28) for γ by letting β_o be the reciprocal mean free path of the gamma rays at the altitude h,

$$\beta_o = \beta_o' \ e^{-kh} \ , \tag{2.30}$$

and by letting the height be measured from the bomb,

$$z_1 = z - h \ ; \tag{2.31}$$

thus,

$$\gamma = \beta_o \ \frac{1-e^{-kz_1}}{kz_1} \ . \tag{2.32}$$

4. CURRENT AND CHARGE DENSITIES

The current and charge densities are related to the polarization such as to satisfy the equation of continuity (Stratton, ref. 7),

$$\vec{J} = \frac{\partial \vec{P}}{\partial t} , \qquad \rho^{*} = - \nabla \cdot \vec{P} . \tag{4.1}$$

4a. Current Density

From the first of Eqs. (4.1) and Eq. (2.27) we obtain

$$\vec{J} = \vec{e}_r \, Ac \, D(ct-r) \, e^{-\gamma r}/r^2 , \tag{4.2}$$

where γ is a function of z defined by Eq. (2.28), A is defined by Eq. (2.29) and D(ct-r) is defined by,

$$D(ct-r) \equiv \begin{cases} 0 & \text{for } ct-r < 0 , \\ \sigma e^{-\sigma(ct-r)} & \text{for } ct-r \geqslant 0 , \end{cases} \tag{4.3}$$

where σ is a function of z defined by Eqs. (2.24) and (2.25). The function D(ct-r) becomes the Dirac delta function when $\sigma \to \infty$, as it should. It is pictured in Fig. 1.

We may check that $\vec{J}$ is the flow of compton electrons by combining Eqs. (4.3) with (2.24) to obtain the coefficient

$$\left(\frac{N}{4\pi} eA'\right)\left[\frac{\beta}{r^2}\right] v \quad , \tag{4.4}$$

where β, the reciprocal mean free path of the gamma rays, is a function of z. Thus, we have the total number of Compton electrons divided by the volume in which they are created times the mean effective velocity of the electrons, which is the appropriate measure of the current density.

The current produced by the Compton electrons is a spherical shell of radially directed current which proceeds outward with the velocity of light. Although the current is radially directed the magnitude is a function of z.

4b. Charge Density

From Eq. (2.27) for $\vec{P}$ it is apparent that the assumption of a point source yields an infinite discontinuity at r = 0. This discontinuity only causes trouble when considering the charge distribution. We, therefore, include the step function $\Delta(r-r_o)$ as a factor that makes the polarization zero inside a sphere of radius r_o which may be thought of as the radius of the bomb. To reduce this factor to unity we merely let $r_o \rightarrow 0$.

Noting that

$$\vec{e}_r = \frac{\rho}{r}\vec{e}_\rho + \frac{z-h}{r}\vec{e}_z \, , \qquad (4.5)$$

we obtain the charge density

$$\rho^* = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[A\,\frac{\rho^2}{r^3}\Delta(r-r_o)\;S(ct-r)e^{-\gamma r}\right] + \frac{\partial}{\partial z}\left[A\,\frac{z-h}{r^3}\Delta(r-r_o)\;S(ct-r)e^{-\gamma r}\right]. \qquad (4.6)$$

Using the derivatives

$$\frac{\partial\lambda}{\partial z} = k\lambda \, ,$$

$$(z-h)\,\frac{\partial\gamma}{\partial z} = -\gamma + \beta \, , \qquad (4.7)$$

as obtained from Eqs. (2.17) and (2.28) where β, the reciprocal mean free path of the gamma rays, is a function of z as given by Eq. (2.6), we obtain

$$\rho^* = -A\,\Delta(r-r_o)\,\frac{e^{-\gamma r}}{r^2}\left\{\left[\beta - \frac{\delta(r-r_o)}{\Delta(r-r_o)}\right]S(ct-r) + \left[1 + k(z-h)\,\frac{ct-r}{r}\right]D(ct-r)\right\}, \qquad (4.8)$$

where S(ct-r) is defined by Eq. (2.23), D(ct-r) by Eq. (4.3), γ by Eq. (2.28), β by Eq. (2.6), and λ by Eq. (2.17).

In addition to the transient charge separation indicated by D(ct-r), there is a permanent charge separation that remains after the passage of the polarization wave, which becomes

$$\rho^* = A \frac{e^{-\gamma r}}{r^2} \left[\delta(r-r_o) - \beta \Delta(r-r_o) \right] \qquad \text{for } t \to \infty . \tag{4.9}$$

This charge distribution is not spherically symmetrical since γ and β are functions of z. The charge distribution consists of a positive charge on the bomb with an equivalent amount of negative charge distributed in space. The net charge is zero since ρ^* is obtained as the divergence of a vector whose normal component vanishes on the sphere at infinity.

5. SCALAR SOLUTION OF MAXWELL'S EQUATIONS

Maxwell's equations for free space (air) in rationalized mks units may be written

$$\text{(I)}\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad \text{(III)}\ \nabla \cdot \vec{B} = 0 ,$$

$$\text{(II)}\ \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_o \vec{J} , \qquad \text{(IV)}\ \nabla \cdot \vec{E} = \rho^*/\epsilon_o , \tag{5.1}$$

where the prescribed sources $\vec{J}$ and ρ^* are given by Eqs. (4.2) and (4.8) for the present problem.

5a. Derivation of the Differential Equation

Due to the complete cylindrical symmetry of source equations (4.2) and (4.8) and the fact that we consider no boundaries, we may introduce a scalar function ψ such that

$$B = \frac{1}{c^2}\frac{\partial \psi}{\partial t} e_\varphi . \tag{5.2}$$

Substituting Eq. (5.2) in the second Maxwell equation (5.1), we obtain

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times \psi \vec{e}_\varphi - \vec{E}) = \mu_o \vec{J} . \tag{5.3}$$

Substituting the first of Eqs. (4.1) into (5.3) and integrating with respect to time, we obtain

$$\vec{E} = \nabla \times \psi \vec{e}_\varphi - \vec{P}/\epsilon_o , \tag{5.4}$$

where we have set the arbitrary function of position equal to zero in order to simultaneously satisfy the second of Eqs. (4.1) and the fourth Maxwell equation (5.1). It may be noted that Eq. (5.2) satisfies the third Maxwell equation (5.1)

Substituting Eqs. (5.4) and (5.2) into the first Maxwell equation (5.1) we obtain the desired differential equation for ψ,

$$\nabla \times \nabla \times \psi \vec{e}_\varphi + \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \vec{e}_\varphi = \nabla \times \vec{P}/\epsilon_o . \tag{5.5}$$

The source term may be obtained from Eqs. (2.26), (4.5), and (4.7),

$$\nabla \times \frac{\vec{P}}{\epsilon_o^{\prime}} = \vec{e}_\varphi \frac{A}{\epsilon_o} \frac{\rho}{r} \frac{e^{-\gamma r}}{r^2} \left[k(ct-r)\, D(ct-r) + \frac{r}{z-h} (\beta-\gamma)\, S(ct-r) \right] . \tag{5.6}$$

Writing out the operator in Eq. (5.5) we finally have,

$$\begin{aligned} &\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho}\right) - \frac{1}{\rho^2} \psi + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \\ &= - \frac{A}{\epsilon_o} \frac{\rho}{r^3} e^{-\gamma r} \left[k(ct-r)\, D(ct-r) + \frac{r}{z-h} (\beta-\gamma)\, S(ct-r) \right] , \end{aligned} \tag{5.7}$$

where $S(ct-r)$ is defined by Eq. (2.23), $D(ct-r)$ by Eq. (4.3), γ by Eq. (2.28), β by Eq. (2.6), and λ by Eq. (2.17).

5b. Boundary Conditions

Since the present investigation is for free space insofar as the electromagnetic field is concerned, we need only specify that the field and all of its derivatives vanish on the sphere at infinity.

6. DERIVATION OF ψ AS AN INTEGRAL

We may construct an integral expression for ψ by using Green's theorem. Taking a time transform of Eq. (5.7) we have

$$\nabla^2 \Psi - \frac{1}{\rho^2} \Psi + \zeta^2 \Psi = F, \tag{6.1}$$

where F is the transform of the source and where

$$\Psi = \int_{-\infty}^{\infty} e^{i\zeta\tau}\, \psi \, d\tau \,, \qquad (6.2)$$

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\zeta\tau}\, \Psi \, d\zeta \,,$$

where τ is the time in units of $1/c$,

$$\tau = ct \,. \qquad (6.3)$$

It has been assumed that

$$\frac{\partial\psi}{\partial\tau} e^{i\zeta\tau} \Big|_{\tau=-\infty}^{\tau=+\infty} = 0 \,, \qquad (6.4)$$

$$\psi\, e^{i\zeta\tau} \Big|_{\tau=-\infty}^{\tau=+\infty} = 0 \,.$$

6a. Green's Theorem

We apply Green's theorem to ψ which satisfies Eq. (6.1) and to the free space Green's function, G, defined by

$$\nabla^2 G - \frac{1}{\rho^2} G + \zeta^2 G = -2\,\delta(z-z')\,\frac{\delta(\rho-\rho')}{\rho} \,, \qquad (6.5)$$

where G is not a function of the cylindrical angle variable, φ. Thus,

$$\int_{\text{all space}} \left[\psi \left(\nabla^2 - \frac{1}{\rho^2} + \zeta^2\right) G - G\left(\nabla^2 - \frac{1}{\rho^2} + \zeta^2\right) \psi \right] dV$$

$$= \int da \left[\psi \frac{\partial G}{\partial r} - G \frac{\partial \psi}{\partial r}\right]_{r\to\infty} = 0 \,. \qquad (6.6)$$

Substituting Eqs. (6.5) and (6.1) into Eq. (6.6) we obtain,

$$\psi(\vec{r}\,') = -\frac{1}{4\pi}\int_{\text{all space}} G\, F\, dV\,. \tag{6.7}$$

6b. Derivation of the Source Transform

We must now evaluate F, the time transform of the source, or, in particular, we must evaluate the time transform of the right side of Eq. (5.7). From Eqs. (2.23) and (4.3) we may obtain,

$$\int_{-\infty}^{\infty} e^{i\zeta\tau} S(\tau-r)\, d\tau = e^{i\zeta r}\left[-\frac{1}{i\zeta} + \frac{1}{i\zeta - v/\lambda c}\right],$$

$$\int_{-\infty}^{\infty} e^{i\zeta\tau}(\tau-r)\, D(\tau-r)\, d\tau = \frac{v}{\lambda c}\;\frac{e^{i\zeta r}}{(i\zeta - v/\lambda c)^2}, \tag{6.8}$$

where it is assumed that Im $\{\zeta\} > 0$. Combining the right side of Eq. (5.7) with Eq. (6.8) we obtain the source transform

$$F = -\frac{A}{\epsilon_0}\,\frac{\rho}{r^3}\, e^{-\gamma r}\, e^{i\zeta r}\left\{\frac{kv}{\lambda c}\,\frac{1}{(i\zeta - v/\lambda c)^2} + \frac{r}{z-h}(\beta-\gamma)\left[-\frac{1}{i\zeta} + \frac{1}{i\zeta - v/\lambda c}\right]\right\}. \tag{6.9}$$

6c. Derivation of the Green's Function

In order to obtain the free space Green's function satisfying Eq. (6.5), we note that the usual free space Green's function (Morse and Feshbach, ref. 3) is

$$\mathcal{G}(\vec{r},\vec{r}\,') = e^{i\zeta R}/R\,, \tag{6.10}$$

which satisfies the differential equation,

$$(\nabla^2 + \zeta^2)\mathcal{G} = -4\pi\,\delta(\vec{r}-\vec{r}\,')\,, \tag{6.11}$$

where

$$R^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi-\varphi') + |z'-z|^2\,. \tag{6.12}$$

Multiplying Eq. (6.11) by $\cos(\varphi-\varphi')$ and integrating with respect to φ from $-\pi$ to $+\pi$, noting that

$$\int_{-\pi}^{\pi} \cos(\varphi-\varphi') \frac{1}{\rho^2} \frac{\partial^2 \mathcal{G}}{\partial\varphi^2} d\varphi = -\frac{1}{\rho^2} \int_{-\pi}^{\pi} \cos\alpha \, \mathcal{G} \, d\alpha , \qquad (6.13)$$

where

$$\alpha = \varphi-\varphi' , \qquad (6.14)$$

we readily see that the solution to Eq. (6.5) is

$$G = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha \cos\alpha \frac{e^{i\zeta R}}{R} . \qquad (6.15)$$

6d. Integral for ψ

Substituting Eqs. (6.15) and (6.9) into Eq. (6.7) we obtain the transform function in integral form,

$$\psi = \frac{A}{8\pi^2\epsilon_o} \int_{\text{all space}} dV \int_{-\pi}^{\pi} d\alpha \cos\alpha \frac{e^{i\zeta R}}{R} \frac{\rho}{r^3} e^{-\gamma r} e^{i\zeta r} \left\{ \frac{kv}{\lambda c} \frac{1}{(i\zeta - v/\lambda c)^2} + \frac{r}{z-h} (\beta-\gamma) \left[-\frac{1}{i\zeta} + \frac{1}{i\zeta - v/\lambda c} \right] \right\} . \qquad (6.16)$$

Integrating with respect to φ and taking the inverse time transform we have the result

$$\psi(t, \rho', z') = \frac{A}{4\pi\epsilon_o} \int_0^{\infty} \rho d\rho \int_{-\infty}^{\infty} dz \int_{-\pi}^{\pi} d\alpha \frac{\rho \cos\alpha}{R} \frac{e^{-\gamma r}}{r^3} \times \left[k(ct-r-R)\, D(ct-r-R) + \frac{r}{z-h} (\beta-\gamma)\, S(ct-r-R) \right] , \qquad (6.17)$$

where R is given by

$$R^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\alpha + |z'-z|^2 , \qquad (6.18)$$

and where r is defined by Eq. (2.2), S(ct-r-R) by Eq. (2.23), D(ct-r-R) by Eq. (4.3), γ by Eq. (2.28), β by Eq. (2.6) and λ by Eq. (2.17).

7. REGION OF INTEGRATION

Since both D(ct-r-R) and S(ct-r-R) appearing in Eq. (6.17) are zero for ct-r-R < 0, the region of integration is bounded by the surface given by

$$\tau - r - R = 0 \,, \tag{7.1}$$

where τ = ct, as before. The surface which is defined by Eq. (7.1) is an allipsoid of revolution with the bomb at one foci and the observer at the other. An ellipsoid of revolution may be defined as that surface for which the sum of the distances from the foci or two fixed points to a point on the surface is a constant; the constant is τ in the present case.

We, thus, introduce prolate spheroidal coordinates by letting

$$r = \frac{1}{2} r' (\xi+\eta) \,,$$
$$R = \frac{1}{2} r' (\xi-\eta) \,, \tag{7.2}$$

where

$$r'^2 = \rho'^2 + (z'-h)^2 \,. \tag{7.3}$$

In particular, if

$$x = \rho \cos \alpha, \quad y = \rho \sin \alpha \,, \tag{7.4}$$

then

$$2\,x = \rho'(1+\xi\eta) - (z'-h) \sin w \sqrt{(\xi^2-1)(1-\eta^2)} \,,$$
$$2\,y = r' \cos w \sqrt{(\xi^2-1)(1-\eta^2)} \,, \tag{7.5}$$
$$2\,(z-h) = (z'-h)\,(1+\xi\eta) + \rho' \sin w \sqrt{(\xi^2-1)\,(1-\eta^2)} \,.$$

The primed coordinates are the coordinates of the observer and are, therefore, constant with respect to the integration, Eq. (6.17). The equation of the ellipsoid of revolution, Eq. (7.1), becomes simply from Eqs. (7.2)

$$\xi = \tau/r' \, . \tag{7.6}$$

To transform the integral given by Eq. (6.17) to prolate spheroidal coordinates, we note that the element of volume is given by

$$dV = \frac{1}{8} r'^3 (\xi^2-\eta^2)\, d\xi\, d\eta dw \, . \tag{7.7}$$

Before transforming to prolate spheroidal coordinates we break up ψ into two parts in order to simplify our expressions,

$$\psi = \frac{A}{4\pi\epsilon_o} (I_1 - I_2) \tag{7.8}$$

where

$$I_1 = \int_{\substack{\text{all}\\ \text{space}}} dV \frac{x}{R} \frac{e^{-\gamma r}}{r^2} f_1 \, ,$$

$$I_2 = \int_{\substack{\text{all}\\ \text{space}}} dV \frac{x}{R} \frac{e^{-\gamma r}}{r^2} f_2 \, , \tag{7.9}$$

where

$$f_1 = k \frac{\tau-r-R}{r} D(\tau-r-R) \, ,$$

$$f_2 = \frac{\gamma-\beta}{z-h} S(\tau-r-R) \, . \tag{7.10}$$

Substituting Eqs. (7.2), (7.4), and (7.5) into Eqs, (7.9) and (7.10), noting that the contribution from the region $\pi/2 \leq w \leq 3\pi/2$ equals the contribution from the region $-\pi/2 \leq w \leq \pi/2$ since the integrand involves only sin w, we have

$$I_1 = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X \frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta} f_1 ,$$

$$I_2 = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X \frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta} f_2 , \qquad (7.11)$$

where

$$f_1 = 2 \frac{k}{\xi+\eta} \left(\frac{\tau}{r} - \xi\right) D(\tau - r\xi) ,$$

$$f_2 = \frac{\gamma-\beta}{Z} S(\tau - r\xi) , \qquad (7.12)$$

where we have dropped the primes and have replaced x by X and z-h by Z in order not to confuse these variables with the point of observation, (z,ρ) .

Collecting other formulas for convenience, we have

$$2X = \rho(1+\xi\eta) - (z-h) \sin w \sqrt{(\xi^2-1)(1-\eta^2)} ,$$

$$2Z = \rho \sin w \sqrt{(\xi^2-1)(1-\eta^2)} + (z-h)(1+\xi\eta) ,$$

$$\gamma = \beta_o \frac{1-e^{-kZ}}{kZ} ,$$

$$\beta = \beta_o e^{-kZ} , \qquad (7.13)$$

$$D(\tau-r\xi) = \sigma e^{-\sigma(\tau-r\xi)} ,$$

$$S(\tau-r\xi) = 1 - e^{-\sigma(\tau-r\xi)} ,$$

$$\sigma = \sigma_o e^{-kZ} ,$$

where β_o is the reciprocal mean free path of the gamma rays at the height h, where σ_o is the apparent effective reciprocal mean free path of the electrons at the height h, and where we need no longer specify that $D(\tau-r\xi)$ and $S(\tau-r\xi)$ are zero for $\tau < r\xi$.

8. INTEGRAL EXPRESSIONS FOR THE FIELD

The electric and magnetic fields may be obtained by differentiating ψ as given by Eqs. (7.8), (7.11), (7.12), and (7.13) according to Eqs. (5.4) and (5.2).

8a. Integral Expression for the Magnetic Field

Taking the time derivation of ψ, Eqs. (7.8), (7.11), (7.12), and (7.13), according to Eq. (5.2), noting that the integrand is zero for $\xi = \tau/r$, we have

$$B_0 = \frac{A}{4\pi\epsilon_0 c}\left[\frac{\partial I_1}{\partial\tau} - \frac{\partial I_2}{\partial\tau}\right] , \tag{8.1}$$

where

$$\frac{\partial I_1}{\partial\tau} = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X \frac{e^{-\gamma r(\xi+\eta)/2}}{\xi + \eta} \frac{\partial f_1}{\partial\tau} ,$$

$$\frac{\partial I_2}{\partial\tau} = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X \frac{e^{-\gamma r(\xi+\eta)/2}}{\xi + \eta} \frac{\partial f_2}{\partial\tau} , \tag{8.2}$$

where

$$\frac{\partial f_1}{\partial\tau} = 2 \frac{k}{\xi+\eta}\left[\frac{1}{r} - (\frac{\tau}{r} - \xi)\sigma\right] D(\tau-r\xi) ,$$

$$\frac{\partial f_2}{\partial\tau} = \frac{\gamma-\beta}{Z} D(\tau-r\xi) . \tag{8.3}$$

8b. Integral Expression for the Electric Field

Using Eq. (5.4) where $\vec{P}$ is given by Eq. (2.27) we obtain,

$$E = -\frac{\partial\psi}{\partial z} + \frac{A}{\epsilon_0} S(ct-r)\rho \frac{e^{-\gamma_1 r}}{r^3} ,$$

$$E_\varphi = 0 , \tag{8.4}$$

$$E_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\psi) + \frac{A}{\epsilon_0} S(ct-r)\ (z-h) \frac{e^{-\gamma_1 r}}{r^3} .$$

The subscript 1 being placed on the γ to distinguish this γ_1 from the γ used in the integrand, Eq. (7.9). To evaluate the z derivative appearing in the first of Eqs. (8.4) and to get the radial component of the electric field we use Eqs. (7.8), (7.11), (7.12), and (7.13) to obtain

$$\frac{\partial\psi}{\partial z} = \frac{A}{4\pi\epsilon_0}\left[\frac{\partial I_1}{\partial z} - \frac{\partial I_2}{\partial z}\right], \tag{8.5}$$

where

$$\frac{\partial I_1}{\partial z} = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X\,\frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta}\, f_1\left[h_z + \frac{1}{f_1}\frac{\partial f_1}{\partial z}\right],$$

$$\frac{\partial I_2}{\partial z} = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X\,\frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta}\, f_2\left[h_z + \frac{1}{f_2}\frac{\partial f_2}{\partial z}\right], \tag{8.6}$$

where again the derivative of the upper limit is zero since f_1 and f_2 vanish for $\xi = \tau/r$, and where

$$h_z = \frac{1}{2X}\frac{\partial}{\partial z}(2X) - \frac{(\xi+\eta)}{2}\frac{\partial}{\partial z}(\gamma r) = -\frac{\sin w\ \sqrt{(\xi^2-1)(1-\eta^2)}}{2X} - \frac{z-h}{2r}(\xi+\eta)\,\gamma + r(\xi+\eta)\,(1+\xi\eta)\,\frac{\gamma-\beta}{4Z}\ . \tag{8.7}$$

and from Eqs. (7.12) we obtain,

$$-\frac{1}{f_1}\frac{\partial f_1}{\partial z} = \frac{\tau}{\tau-r\xi}\,\frac{z-h}{r^2} - \frac{z-h}{r}\,\sigma\xi + \frac{1}{2}\,k\,(1+\xi\eta)\big[(1 - \sigma(\tau - r\,\xi))\big]$$

$$-\frac{1}{f_2}\frac{\partial f_2}{\partial z} = \frac{1+\xi\eta}{Z} - \frac{1}{2}\,k\,\frac{\beta}{\gamma-\beta}\,(1+\xi\eta) + \left[\frac{z-h}{r} - \frac{1}{2}\,k(\tau - r\xi)(1+\xi\eta)\right]\frac{D(\tau - r\xi)}{S(\tau - r\xi)}\ . \tag{8.8}$$

We may similarly obtain the z component of the electric field by using Eqs. (7.8), (7.11), (7.12), and (7.13) to evaluate the derivative appearing in the last of Eqs. (8.4); thus,

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\psi) = \frac{A}{4\pi\epsilon_0}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_1) - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_2)\right], \qquad (8.9)$$

where

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_1) = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X\,\frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta}\, f_1\left[h_\rho + \frac{1}{f_1}\frac{\partial f_1}{\partial\rho}\right],$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_2) = \int_{-\pi/2}^{\pi/2} dw \int_{-1}^{1} d\eta \int_{1}^{\tau/r} d\xi \quad 2X\,\frac{e^{-\gamma r(\xi+\eta)/2}}{\xi+\eta}\, f_2\left[h_\rho + \frac{1}{f_2}\frac{\partial f_2}{\partial\rho}\right], \qquad (8.10)$$

where

$$h_\rho = \frac{1}{\rho} + \frac{1}{2X}\frac{\partial}{\partial\rho}(2X) - \frac{(\xi+\eta)}{2}\frac{\partial}{\partial\rho}(\gamma r) = \frac{1}{\rho}$$
$$+ \frac{1}{2}\frac{1+\xi\eta}{X} - \frac{1}{2}\frac{\rho}{r}(\xi+\eta)\,\gamma + r\,\frac{\gamma-\beta}{4Z}(\xi+\eta)\sin w\,\sqrt{(\xi^2-1)(1-\eta^2)}, \qquad (8.11)$$

and from Eqs. (7.12) we obtain,

$$\frac{1}{f_1}\frac{df_1}{d\rho} = -\frac{\tau\rho}{r^2(\tau-\xi r)} + \frac{\rho}{r}\,\sigma\,\xi - \frac{1}{2}\,k\sin w\,\sqrt{(\xi^2-1)(1-\eta^2)}\;(1+r\sigma\xi),$$

$$\frac{1}{f_2}\frac{df_2}{d\rho} = \sin w\,\sqrt{(\xi^2-1)(1-\eta^2)}\left[\frac{1}{2}\,k\,\frac{\beta}{\gamma-\beta} - \frac{1}{Z}\right] \qquad (8.12)$$
$$+ \xi\,\frac{D(\tau-r\xi)}{S(\tau-r\xi)}\left[\frac{1}{2}\,kr\sin w\,\sqrt{(\xi^2-1)(1-\eta^2)} - \frac{\rho}{r}\right].$$

The electric field in integral form may, thus, be obtained as the indicated combinations of the equations of this section.

9. EXPRESSIONS FOR COMPUTATION

Since the present work is a preliminary investigation, there has been no attempt to obtain analytical results. Instead a few cases will be sampled by numerical integration.

There are six fundamental physical parameters,

$$\begin{aligned} &\rho, \quad z_1, \quad \tau, \\ &\beta_o, \quad \sigma_o, \quad k, \end{aligned} \tag{9.1}$$

where ρ is the horizontal distance of the observer from the position of the bomb, z_1 is the vertical position of the observer as measured from the position of the bomb,

$$z_1 \equiv z - h \quad , \tag{9.2}$$

τ is the time in units of $1/c$

$$\tau = ct \quad , \tag{9.3}$$

β_o is the reciprocal mean free path of the gamma rays at the height h above sea level, σ_o is the apparent effective reciprocal mean free path of the Compton electrons at the height h above sea level,

$$\sigma_o \equiv v/c\lambda_o \quad , \tag{9.4}$$

where λ_o is defined by Eq. (3.5), and k is the reciprocal scale height of the earth's atmosphere.

9a. Expression for ψ

From Eqs. (7.8), (7.11), (7.12), and (7.13) we may write

$$\frac{4\pi\epsilon_o}{A}\psi = I_1 - I_2 \quad , \tag{9.5}$$

where

$$I_{1,2} = \int_{-\pi/2}^{\pi/2} dw \int_0^{\pi} dv \int_0^{\cosh^{-1}a} du \quad Gf_{1,2} , \tag{9.6}$$

where we have introduced new variables of integration u and v such that

$$\xi = \cosh u, \qquad \eta = \cos v , \tag{9.7}$$

in order to keep the integrand finite everywhere. The constant (with respect to the integration), a, is

$$a = \tau/r . \tag{9.8}$$

The function G which is common to both I_1 and I_2 is given by

$$G = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi+\eta} \; 2X \, g , \tag{9.9}$$

where

$$\begin{aligned} 2X &= \rho(1+\xi\eta) - z_1 \sin w \sqrt{(\xi^2-1)(1-\eta^2)} , \\ 2Z &= \rho \sin w \sqrt{(\xi^2-1)(1-\eta^2)} + z_1 (1+\xi\eta) , \\ g &= \exp \left[- b_1 \frac{1-p}{kZ} (\xi+\eta) \right] , \end{aligned} \tag{9.10}$$

where

$$\begin{aligned} p &= e^{-kZ} \\ b_1 &= \frac{1}{2} \beta_o r , \\ z_1 &= z-h . \end{aligned} \tag{9.11}$$

The functions f_1 and f_2, Eq. (9.6) which yield I_1 and I_2, become

$$\begin{aligned} f_1 &= \frac{2k}{r} \frac{b_2 p (a-\xi)}{\xi+\eta} f , \\ f_2 &= \frac{2b_1}{r} \frac{1 - p(1+kZ)}{kZ^2} (1-f) , \end{aligned} \tag{9.12}$$

where

$$f = e^{-b_2 p(a-\xi)} ,$$

$$b_2 = \sigma_o r \quad . \tag{9.13}$$

9b. Expression for B_φ

From Eqs. (8.1), (8.2), and (8.3) and the equations of this section we obtain

$$B_\varphi = \frac{A}{4\pi\epsilon_o c}\left[\frac{\partial I_1}{\partial\tau} - \frac{\partial I_2}{\partial\tau}\right] , \tag{9.14}$$

where

$$\frac{\partial I_{1,2}}{\partial\tau} = \int_{-\pi/2}^{\pi/2} dw \int_0^{\pi} dv \int_0^{\cosh^{-1}a} du \quad G\,\frac{\partial f_{1,2}}{\partial\tau} \tag{9.15}$$

where

$$\frac{\partial f_1}{\partial\tau} = \frac{2kb_2}{r^2}\,\frac{1-b_2 p(a-\xi)}{\xi+\eta}\,pf ,$$

$$\frac{\partial f_2}{\partial\tau} = \frac{2b_1 b_2}{r^2}\,\frac{1-p(1+kZ)}{kZ^2}\,pf \quad . \tag{9.16}$$

9c. Expressions for the Electric Field

We rewrite Eqs. (8.4), (8.5), and (8.9) to obtain,

$$E_\rho = \frac{A}{4\pi\epsilon_o}\left\{-\frac{\partial I_1}{\partial z} + \frac{\partial I_2}{\partial z} + 4\pi\,(1-f_o)\,\frac{\rho}{r^3}\,g_o\right\} ,$$

$$E_\varphi = 0 , \tag{9.17}$$

$$E_z = \frac{A}{4\pi\epsilon_o}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_1) - \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho I_2) + 4\pi\,(1-f_o)\,\frac{z_1}{r^3}\,g_o\right\} ,$$

where

$$f_o = e^{-b_2 p_o (a-1)} ,$$

$$p_o = e^{-kz_1} , \qquad (9.18)$$

$$g_o = \exp\left\{-2b_1 \frac{1-p_o}{kz_1}\right\} .$$

For the radial component of the electric field, we obtain from Eqs. (8.6), (8.7), and (8.8),

$$\frac{\partial I_{1,2}}{\partial z} = \int_{-\pi/2}^{\pi/2} dw \int_0^{\pi} dv \int_0^{\cosh^{-1} a} du \; G f_{1,2} \left[h_z + \frac{1}{f_{1,2}} \frac{\partial f_{1,2}}{\partial z} \right] , \qquad (9.19)$$

where G is defined by Eq. (9.9), $f_{1,2}$ by Eq. (9.12) ,

$$h_z = - \frac{\sin w \sqrt{(\xi^2-1)(1-\eta^2)}}{2X} - \frac{z_1 b_1}{r^2} \frac{1-p}{kZ} (\xi+\eta)$$
$$+ \frac{1}{2} b_1 \frac{1-p(1+kZ)}{kZ^2} (\xi+\eta)(1+\xi\eta) , \qquad (9.20)$$

and

$$- \frac{1}{f_1} \frac{\partial f_1}{\partial z} = \frac{z_1}{r^2} \left[\frac{a}{a-\xi} - b_2 p \xi \right] + \frac{1}{2} k \, (1+\xi\eta)(1+b_2 p\xi) ,$$
$$- \frac{1}{f_2} \frac{\partial f_2}{\partial z} = \frac{1-p(1+kZ+\frac{1}{2}k^2Z^2)}{1-p(1+kZ)} \frac{1+\xi\eta}{Z} + b_2 \left[\frac{z_1}{r^2} - \frac{1}{2} k \, (1+\xi\eta) \right] \xi p \frac{f}{1-f} , \qquad (9.21)$$

where f is given by Eq. (9.13) and p by Eq. (9.11).

For the z component of the electric field we obtain from Eqs. (9.17), (8.10), (8.11), and (8.12) ,

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho I_{1,2}) = \int_{-\pi/2}^{\pi/2} dw \int_0^{\pi} dv \int_0^{\cosh^{-1} a} du \; G f_{1,2} \left[h_\rho + \frac{1}{f_{1,2}} \frac{\partial f_{1,2}}{\partial \rho} \right] , \qquad (9.22)$$

where

$$h_\rho = \frac{1}{\rho} + \frac{1+\xi\eta}{2X} - \frac{\rho b_1}{r^2}\frac{1-p}{kZ}(\xi+\eta)$$

$$+ \frac{1}{2} b_1 \frac{1-p(1+kZ)}{kZ^2}(\xi+\eta) \sin w \sqrt{(\xi^2-1)(1-\eta^2)} \quad , \qquad (9.23)$$

and

$$\frac{1}{f_1}\frac{\partial f_1}{\partial \rho} = \frac{\rho}{r^2}\left(-\frac{a}{a-\xi} + b_2 p\xi\right) - \frac{1}{2} k\,(1+b_2 p\xi) \sin w \sqrt{(\xi^2-1)(1-\eta^2)} \quad ,$$

$$\frac{1}{f_2}\frac{\partial f_2}{\partial \rho} = -\frac{1-p(1+kZ+\frac{1}{2}k^2Z^2)}{1-p(1+kZ)}\,\frac{\sin w \sqrt{(\xi^2-1)(1-\eta^2)}}{Z} \qquad (9.24)$$

$$+ b_2\left[\frac{1}{2} k \sin w \sqrt{(\xi^2-1)(1-\eta^2)} - \frac{\rho}{r^2}\right]\xi\, p\,\frac{f}{1-f} \quad ,$$

where f is given by Eq. (9.13) and p by (9.11).

10. STATIC FIELDS

After the polarization wave has passed to infinity a static charge distribution remains behind, Eq. (4.9), giving rise to a static electric field. This static charge distribution is of interest as a limiting case of the general results. Because the bomb ionizes the air in nearly a spherical region, plasma oscillations, produced either by the flow of charges left by the polarization wave or by the transient field reacting back upon the conducting air, are expected to be relatively small as compared with a ground shot (ref. 8).

10a. Static Value of ψ

The static value of ψ is obtained by letting $\tau \to \infty$ in Eq. (6.17). When $\tau \to \infty$ the polarization wave, Eq. (2.27), has passed to infinity. From Eqs. (2.23) and (4.3) we have

$$\left.\begin{aligned} (\tau-r-R)\, D(\tau-r-R) &= 0 \\ S(t-r-R) &= 1 \end{aligned}\right\} \quad \text{for } \tau \to \infty . \qquad (10.1)$$

Substituting Eqs. (10.1) into Eq. (6.17) with a change in notation we have,

$$\psi_s(\rho, z_1) = \frac{A}{4\pi\epsilon_o} \int_0^{\infty} \lambda \, d\lambda \int_{-\infty}^{\infty} d\xi \int_{-\pi}^{\pi} d\alpha \, \frac{\lambda \cos\alpha}{R} \, \frac{e^{-\gamma r}}{r^2} \, \frac{\beta-\gamma}{\xi} \quad , \qquad (10.2)$$

where ρ' has been replaced by ρ, z' by z, ρ by λ, and z by $(h+\xi/k)$ and where from Eqs. (6.18), (2.2), (2.6), and (2.28),

$$\begin{aligned} R^2 &= \rho^2 - 2\rho\lambda\cos\alpha + (z_1-\xi/k)^2 + \lambda^2 \quad , \\ r^2 &= \lambda^2 + \xi^2/k^2 \quad , \\ \beta &= \beta_o e^{-\xi} \\ \gamma &= \beta_o\,(1-e^{-\xi})/\xi \quad . \end{aligned} \qquad (10.3)$$

The integration with respect to α in Eq. (10.2) may be performed in terms of complete elliptic integrals. Using the result (ref. 8, p. 33, Eq. (8.5))

$$\int_{-\pi}^{\pi} d\alpha \, \frac{\cos\alpha}{R} = \frac{4}{x\sqrt{\rho\lambda}} \left[(1-x^2/2)\, K(x) - E(x) \right] , \qquad (10.4)$$

where K is the complete elliptic integral of the first kind and E is the complete elliptic integral of the second kind (Franklin, ref. 1), where

$$x^2 = \frac{4\rho\lambda}{(\rho+\lambda)^2 + (z_1-\xi/k)^2} \quad , \qquad (10.5)$$

we may write Eq. (10.2) as

$$\psi_s = -\frac{A\beta_o}{\pi\epsilon_o\sqrt{\rho}} \int_{-\infty}^{\infty} d\xi \, p_2 \int_0^{\infty} d\lambda \quad \lambda^{3/2} \, \frac{e^{-\gamma r}}{r^2} \, H(x), \qquad (10.6)$$

where

$$H(x) = \frac{1}{x} \left[(1-\tfrac{1}{2}x^2)\, K(x) - E(x) \right] , \qquad (10.7)$$

and from Eq. (10.3)

$$p_2 = \left[1 - (1+\xi)\, e^{-\xi}\right] \Big/ \xi^2 \qquad (10.8)$$

For numerical work in order to have a finite range of integration, we introduce the change of variables,

$$\zeta = \frac{\lambda}{\lambda + \rho} \;, \quad d\lambda = \frac{\rho}{(1-\zeta)^2}\, d\zeta \;,$$
$$s = \tan^{-1}\xi \;, \quad d\xi = ds/\cos^2 s \;. \qquad (10.9)$$

Substituting Eqs. (10.9) into Eq. (10.6) we obtain

$$\Psi_s = -\frac{A\beta_o \rho^2}{\pi\epsilon_o} \int_{-\pi/2}^{\pi/2} ds\; \frac{p_2}{\cos^2 s} \int_0^1 d\zeta\; \frac{\zeta^{3/2}}{(1-\zeta)^{7/2}}\, \frac{e^{-\gamma r}}{r^2}\, H \;, \qquad (10.10)$$

where H is defined by Eq. (10.7), p_2 by Eq. (10.8), R, r, γ by Eqs. (10.3) and where from Eq. (10.5),

$$x^2 = \frac{4\zeta\,(1-\zeta)}{1 + (1-\zeta)^2 \left[z_1/p - (\tan^{-1}s)/k\rho\right]^2} \;. \qquad (10.11)$$

10b. Static Electric Field, z Component

The z component of the static electric field may be obtained by substituting Ψ_s, Eq. (10.6), for Ψ in the last of Eq. (8.4), and noting Eq. (10.1). Thus

$$E_{zs} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Psi_s) + \frac{A}{\epsilon_o}(z-h)\,\frac{e^{-\gamma_1 r}}{r^3} \;, \qquad (10.12)$$

where from Eq. (2.28)

$$\gamma_1 = \beta_o(1-e^{-kz_1})/z_1 \;, \qquad (10.13)$$

and where by differentiating Eq. (10.6) we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\psi_s) = -\frac{A\beta_o}{2\pi\epsilon_o\rho^{3/2}}\int_{-\infty}^{\infty} d\xi \quad p_2 \int_0^{\infty} d\lambda \quad \lambda^{3/2} \; \frac{e^{-\gamma r}}{r^2} \; I , \qquad (10.14)$$

where

$$I = H + 2\rho \frac{dH}{dx}\frac{\partial x}{\partial\rho} . \qquad (10.15)$$

Using the relations

$$\frac{dK}{dx} = \frac{1}{x}\left[\frac{E}{1-x^2} - K\right] ,$$

$$\frac{dE}{dx} = \frac{1}{x}(E - K) , \qquad (10.16)$$

$$\frac{\partial x}{\partial\rho} = \frac{x}{2\rho}\left[1 - \frac{\lambda+\rho}{2\lambda}x^2\right] ,$$

Eq. (10.15) becomes

$$I = \frac{x}{4\lambda(1-x^2)}\left\{\lambda x^2 E + 2\rho\left[(1-x^2) K - (1-x^2/2) E\right]\right\} . \qquad (10.17)$$

For computational purposes we may again make the change of variables given by Eqs. (10.9). The z component of the static electric field is, thus, given by Eqs. (10.12), (10.14), (10.17), (10.8), (10.5) and Eq. (10.3).

10c. Static Electric Field, ρ Component

The radial component of the static electric field may be obtained by substituting ψ_s, Eq. (10.6), for ψ in the first of Eq. (8.4), noting Eq. (10.1). Thus

$$E_{\rho s} = -\frac{\partial\psi_s}{\partial z} + \frac{A}{\epsilon_o}\;\rho\;\frac{e^{-\gamma_1 r}}{r^3} , \qquad (10.18)$$

where

$$-\frac{\partial\psi_s}{\partial z} = \frac{A\beta_0}{\pi\epsilon_0\sqrt{\rho}} \int_{-\infty}^{\infty} d\xi \quad p_2 \int_0^{\infty} d\lambda \quad \lambda^{3/2} \; \frac{e^{-\gamma r}}{r^2} \; \frac{dH}{dx} \; \frac{\partial x}{\partial z} \quad , \tag{10.19}$$

where from Eq. (10.16) and

$$\frac{\partial x}{\partial z} = -\frac{x^3}{4\rho\lambda} (z_1 - \xi/k) \quad , \tag{10.20}$$

we have

$$\frac{\partial x}{\partial z} \; \frac{dH}{dx} = \frac{x(z_1 - \xi/k)}{4\rho\lambda\,(1-x^2)} \left[(1-x^2)\,K - (1-x^2/2)\,E \right] . \tag{10.21}$$

We may again make the change of variables, Eq. (10.9), for computation. The radial component of the static electric field is obtained by combining Eqs. (10.18), (10.19), (10.21), (10.8), (10.5), and Eq. (10.3).

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