Classical Interpretation of Quantum Mechanics*

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Following de Broglie, Bohm, and others, it is assumed that quantum mechanics may be interpreted causally and that the ψ function plays the role of a generating function for particle trajectories. By arguing that the ψ function should not be interpreted as a probability amplitude, a new method for generating particle trajectories is postulated. The four-momentum of a scalar particle is assumed to be given as the gradient of an unspecified function $F(\psi)$, where ψ is a pure real solution of the Klein-Gordon equation. Since the location of a particle is determined solely by its trajectory, the probability distribution differs from $\psi\psi^*$; and therefore, ordinary experimental results differing from the traditional theory may, in principle, be predicted. Particle motion and trajectories are discussed for three examples: a free particle, a particle in a box, and the double slit.

I. INTRODUCTION

An excellent review of various causal formulations of quantum mechanics has been presented by Freistadt.[1] The theory presented here, in common with previous causal interpretations, is in opposition to the traditional views of Bohr[2] and Heisenberg.[3] Fundamentally, the theory presented here differs from previous causal interpretations of quantum mechanics by, first, not interpreting ψ as a probability amplitude and, second, by choosing only pure real representations for ψ. As a consequence, a new method for generating trajectories from the ψ function is obtained. The trajectories obtained differ considerably from the trajectories obtained in previous causal theories. Because the probability distribution determined from the trajectories is not $\psi\psi^*$, the theory presented here also differs quite markedly from previous causal theories by permitting the possibility of an ordinary experimental test of its validity. It is not, consequently, equivalent to, or isomorphic with, traditional interpretations of quantum mechanics.

II. DIFFICULTIES ARISING FROM THE PROBABILITY INTERPRETATION OF ψ

Here it is felt that the attempt to make the ψ function perform double duty as both a wave function and a probability amplitude leads to a number of difficulties which may be resolved by relinquishing the probability interpretation. Others have also questioned the Born probability interpretation.[4]

1. Wave-Particle Paradox

The macroscopic concept of a wave involves the notion of a function which describes a physical property continuously throughout space. The concept of a particle, on the other hand, implies that all of the properties of the particle are localized in a very small region of space (which may be represented by a point). The wave-particle paradox then arises when the question is asked: How can a single propagated entity exhibit two such completely dissimilar properties?

Assuming that wave and particle characteristics are not intrinsically inextricable from each other, the resolution of the paradox may be sought in one of four possible ways: (1) The entity is a particle which carries a pilot wave along with it as envisioned by de Broglie.[5] (2) It is a wave which possesses a traveling singularity which looks and acts like a particle, as suggested by Einstein and developed by Petiau.[6] (3) It is, in fact, only a wave which is mistakenly interpreted as a particle on some occasions. (4) It is, in fact, only a particle (or collection of particles) which is mistakenly interpreted as a wave on some occasions.

Here, (1), (2), and (3) are rejected because of the peculiar or impossible way a wave (pilot wave, wave with a singularity, bunched wave, or a wave packet) would have to behave to satisfy the observations concerning emission and absorption. A particle is emitted from a highly localized region in space and eventually comes to rest (is absorbed) in a highly localized region in space without loss of rest mass. A classical wave, on the other hand, when emitted from a highly localized region in space continues to diffuse outward and can never again be concentrated down to as small a region in space when absorbed. This behavior is characteristic of solutions to wave equations with commensurable boundary conditions and is in agreement with the second law of thermodynamics and ray optics. If it is assumed that a wave can, in fact, collapse down into

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[1] H. Freistadt, Nuovo cimento Suppl. **10**, 1–70 (1957).

[2] N. Bohr, *Atomic Physics and Human Knowledge* (John Wiley & Sons, Inc., New York, 1958).

[3] W. Heisenberg, *Physics and Philosophy* (Harper and Brothers, New York, 1958).

[4] A. George, editor, *Louis de Broglie, Physicien et Penseur* (Paris, 1953), especially the contributions of (i) A. Einstein; (ii) E. Schrödinger; (iii) W. Pauli; (iv) L. Rosenfeld; (v) W. M. Elsasser; (vi) A. March; (vii) H. Reichenbach; (viii) M. A. Tonnelat. This volume includes a bibliography of de Broglie's works to 1953.

[5] L. de Broglie, *Une Tentative d'Interprétation Causale et non Linéaire de la Mécanique Ondulatoire* (Gauthier-Villars, Paris, 1956).

[6] G. Petiau, Compt. rend. **239**, 344–346 (1954).

a localized region when absorbed (called a "quantum jump" by Schrödinger[7]), then such a process will not only apparently violate the second law of thermodynamics but will also involve velocities greater than the velocity of light.[8,9] Consequently, the resolution of the wave-particle paradox chosen here is: (4) The propagated entity is always a particle (or collection of particles) which may be mistakenly interpreted as a wave on some occasions.

Since primitive observations or measurements of submicroscopic phenomena involve only processes of emission and absorption of elementary particles, it would seem that the unobserved phenomena intervening between emission and absorption should be most appropriately described in terms of particles only.

Since the ψ function is assumed here to play the single role of a generating function for particle trajectories, no physical wave is assumed to exist; and the wave-particle paradox is resolved.

It should be noted that the solution of a wave equation with appropriate boundary condition is a wave only in the mathematical sense; and before it can represent a physical wave, which is defined by observations and operations in the laboratory, the mathematical wave function must be identified with some physically observable property in the laboratory such as the displacement of a rope, the density of air, the component of electric intensity, etc. Here, the mathematical wave properties of the ψ function are not identified with any physically observable property, and no physical wave is assumed, in fact, to exist. It is felt here that particle probability distributions are not *precisely* wavelike and that more careful experiments may eventually be able to show this.

Since ψ is treated here solely as a generating function for particle trajectories, it does not necessarily imply the existence of a physical property defined throughout space and, thus, does not necessarily imply the existence of a physical wave. In precisely the same way, the generating function obtained from Hamilton's partial differential equation in classical mechanics does not necessarily imply the existence of an actual physical property defined throughout space.

2. Double-Slit Paradox

The error committed in the usual statement of the double-slit paradox is the assumption that the slit through which the particle does not pass will not affect the trajectory of the particle. The paradox may be resolved by assuming that the particle is, in fact, strongly influenced by both slits while passing through only one[10] (compare the trajectories found here, Sec. VIII).

3. Wrong Boundary Conditions

The boundary conditions placed on ψ that it be continuous, finite, single-valued, and vanish at infinity specify a proper wave function. But if ψ were, indeed, a probability amplitude, it would not have to be continuous, it would not have to remain finite, and it would not have to be single-valued. It would appear then that the wrong boundary conditions are chosen in traditional quantum mechanics, since the very solution which is claimed to be sought is rejected at the outset. The difficulty is removed by the theory presented here by relinquishing the probability interpretation of ψ.

4. Superposition Paradox

Although traditional quantum theory fails to give a causal explanation of individual events, it is felt that probability distributions can be explained causally by employing the "superposition postulate." If the ψ function is interpreted as a classical wave function, it may clearly be treated as a causal phenomenon which obeys Huygen's superposition of wavelets; but it does not appear to be possible to attach the same causal description to ψ if it is interpreted as a probability amplitude. For example, in the classical language of the double-slit experiment, causality is established by saying that the wavelet from one slit interferes with the wavelet from the other slit to produce or *cause* a particular intensity at a given point. Correspondingly, in the language of probability it would have to be said that the probability of finding a particle in a given element of space and time is caused by the probabilities elsewhere in space (in particular, is caused by the probability amplitude functions associated with each slit). But the probabilities of finding the particle in other positions are probabilities associated with events that never happened, and events that never happened could not be said to *cause* the event observed. Another example of the superposition paradox is Schrödinger's[11] "cat in a box paradox."

As a result of difficulties pointed out by Einstein, Rosen, and Podolsky,[12] a number of devices have been introduced in an attempt to resolve the superposition paradox: A particle is not confined to a point but is distributed throughout all space and experiences all possible events; but such superpositions over all available quantum states are never actually observed, because the act of measurement forces the system into just one quantum state. It is not meaningful to inquire into the nature of a system when it is not being observed; etc. Although these devices may appear to resolve the superposition paradox, they are, nevertheless, neglected here because they do not seem to permit, in principle, the possibility of either empirical verification or refutation.

[7] E. Schrödinger, Naturwissenschaften **23**, 483 (1935).
[8] L. Jánossy, Ann. Physik **11**, 323 (1955).
[9] D. Bohm, Phys. Rev. **85**, 180 (1952).
[10] D. Bohm, *Causality and Chance in Modern Physics* (D. Van Nostrand Company, Princeton, New Jersey, 1957).
[11] E. Schrödinger, Brit. J. Phil. Sci. **3**, 109 (1952).
[12] A. Einstein, N. Rosen, and B. Podolsky, Phys. Rev. **47**, 777 (1935).

The theory presented here resolves the paradox by not attempting to give a causal construct in terms of probabilities—ψ is not given a probability interpretation.

5. Question of the Aether

If light is conceived of as a causal wave in a material medium, an "aether" is needed for radiant energy to be transmitted across a void; but the Michelson-Morley experiment makes the notion of an aether untenable. In an attempt to minimize this problem, it is frequently said that an electromagnetic wave "just exists in empty space" and it is either "meaningless" or "unnecessary" to speak of an aether. Here, the difficulty is resolved by assuming that energy and momentum are transmitted by particles and not waves. The electromagnetic field is interpreted as having no direct physical significance; the field presumably determines the position and polarization of a photon as a function of time (a view shared by Nagy[13]).

III. THE IMAGINARY NUMBER i

The imaginary number i, which appears in operators, in Schrödinger's time-dependent equation, and throughout traditional quantum theory, seems to be an integral part of the traditional quantum theory. Since it should be trivial to convert complex expressions to pure real expressions and since ψ must have the same physical information whether expressed in a pure real or in a complex form, the existence of the imaginary number i in the traditional theory is disturbing. Landé[14] has shown that the occurrence of the i stems from the superposition of probability amplitudes.

Here, ψ is not given a probability interpretation; consequently, it is possible to assume pure real expressions for ψ and to assume only the formalisms that permit such pure real expressions.

IV. SOME OBJECTIONS TO THE DE BROGLIE-BOHM THEORY

The theory proposed in this paper is in agreement with the causal theories of de Broglie[5] and Bohm[9] in that the ψ function is used to generate particle trajectories. The actual prescription and the trajectories, however, are considerably different. The theory proposed here does not appear to suffer from certain defects in the de Broglie-Bohm theory.

The formulation of de Broglie and Bohm gives the momentum as $\mathbf{p}=\nabla S$, where $\psi=R\exp(iS/\hbar)$. Consequently, if ψ is pure real, the particle remains at rest for all time. Since there appears to be a question as to the necessity for accepting only certain complex representations, there would seem to be some ambiguity as to whether or not a particle is at rest.

According to their solution, a particle in a box with non-zero energy remains at rest for all time, even when the walls are removed to infinity. There is, thus, an apparent failure to yield satisfactory correspondence with classical macroscopic observations.

[13] K. Nagy, Acta Phys. Acad. Sci. Hung. **4**, 327 (1955).

[14] A. Landé, Phys. Rev. **108**, 891 (1957).

V. DERIVATION OF THE CLASSICAL INTERPRETATION

A quantum-mechanical theory for a scalar particle which is free of the imaginary number i may be obtained by generalizing from the de Broglie wave.

1. The de Broglie Wave[15]

Accepting the Planck-Einstein frequency condition, a physical system may be characterized by an angular frequency ω, given by

$$\omega=E/\hbar, \tag{1}$$

where E is the total energy of the system and $\hbar$ is Planck's constant. For a system or particle whose total energy is specified in terms of its rest mass, the frequency is

$$\omega=mc^2/\hbar. \tag{2}$$

This frequency may be thought of in analogy with mass, the evidence for both ω and for m being deduced from the dynamical behavior of the system. Some characteristic of the system ψ may be represented by the function

$$\psi=\psi_0\sin\omega t, \tag{3}$$

where real, rather than the usual complex, representation has been chosen. If the system or particle is translated with a uniform velocity $\mathbf{v}$ a stationary observer will view the characteristic as

$$\psi=\psi_0\sin[\omega(t-\mathbf{v}\cdot\mathbf{r}/c^2)/\gamma],\quad \gamma=(1-v^2/c^2)^{\frac{1}{2}}, \tag{4}$$

according to the special-relativity time transformation where ψ_0 has been assumed to be invariant. Using Eq. (2) and the relations

$$\mathbf{p}=m\mathbf{v}/\gamma,\quad E=mc^2/\gamma, \tag{5}$$

Eq. (4) may be written in the form

$$\psi=\psi_0\sin[(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar], \tag{6}$$

where, for the present case, $\mathbf{p}$ and E are constants. This result, Eq. (6), represents a traveling wave with the phase velocity c^2/v and the group velocity v.

2. Klein-Gordon Equation

Since Eq. (6) is a wave function, the results may be generalized by considering solutions of a wave equation that satisfy appropriate boundary conditions. This postulated generalization introduces the concept of boundaries acting on and determining the motion of particles. It might be speculated that the boundaries set up some sort of static field with which the particles

[15] L. de Broglie, Ann. Physik **3**, 22 (1925).

have a phase-dependent interaction. The simplest relativistically invariant wave equation which is satisfied by pure real sine or cosine representations is the Klein-Gordon equation

$$\nabla^2\psi - \partial^2\psi/\partial(ct)^2 = (mc/\hbar)^2\psi, \tag{7}$$

where the constant term on the right has been included so that when the solution, Eq. (6), is substituted into Eq. (7) the relation

$$-p^2 + E^2/c^2 = m^2c^2 \tag{8}$$

is obtained in conformity with Eqs. (5).

3. Specification of Trajectories

It is postulated that the trajectories of a relativistic scalar particle may be generated from the known eigenvalue-eigenfunction solution of Eq. (7); thus,

$$\mathbf{p} = \mathbf{p}[A,B,C,\psi(\mathbf{r},t)], \quad E = E[A,B,C,\psi(\mathbf{r},t)], \tag{9}$$

where A, B, and C are the three eigenvalues and $\psi(\mathbf{r},t)$ is the known eigenfunction. It is assumed that the position of the particle as a function of time may be obtained by dividing these four simultaneous first-order differential equations two at a time and integrating. This process will introduce three constants of integration, apart from the initial time which may be taken to be zero. These three constants together with the three eigenvalues, A, B, and C, specify the requisite (in the Newtonian sense) six constants of the motion.

There appears to be no way the functions in Eq. (9) can be chosen uniquely. The simplest specification (for a scalar particle) appears to be given in terms of a single scalar function $F(\psi)$ by letting $\mathbf{p}$ and E form the four-vector

$$\mathbf{p} = F'\nabla\psi, \quad E = -F'\partial\psi/\partial t, \tag{10}$$

where the primes denote differentiation with respect to ψ. This choice has been indicated by the differential Eq. (7). Using Eqs. (10) and a number of identities, Eq. (7) may be rewritten in the form

$$\hbar^2F'^2(c^2\nabla\cdot\mathbf{p} + \partial E/\partial t) + \hbar^2F''(-c^2p^2 + E^2) = m^2c^4F'^3\psi. \tag{11}$$

In the macroscopic limit as $\hbar \to 0$, Eq. (11) should reduce to Eq. (8), or

$$\hbar^2F''/F'^3\psi \sim 1, \tag{12}$$

where the first term on the left of Eq. (11), which may be compared with the conservation of mass-energy expression for a relativistic fluid,[16] vanishes as $\hbar$. Since Eqs. (10) are to be divided two at a time to obtain integrable expressions, the explicit form of $F(\psi)$ (which can always be obtained after integration) is a matter of indifference.

[16] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1942), p. 125, Eq. (8.13).

4. Interpretation of ψ

The theory presented here does not yield any simple identification of the ψ function with any physically observable property such as is provided by the traditional probability-amplitude interpretation (which has been rejected here). The mathematical wave properties of the ψ function are not identified here with any physically observable wave properties. In fact, it is felt that no physical wave actually exists and that, eventually, experiments may be able to demonstrate this by discovering that the particle probability distribution is not given precisely by the traditional expression $\psi\psi^*$.

The mathematical wave function ψ has been interpreted here solely as a generating function for particle trajectories; consequently, there is no need to assume that any actual physically observable property exists throughout all space or that a physical wave actually exists. If a physically significant interpretation of ψ is desired it can, in principle, always be obtained from Eqs. (10), (11), and (12); since these equations relate the generating function ψ to the observables. However, such a physical interpretation of the ψ function may be superfluous (compare the Hamilton-Jacobi generating function in classical mechanics). Even though it is true that the ψ function, as well as the Hamilton-Jacobi function, contains all of the pertinent physical information and is intimately dependent upon physical conditions, it does not appear to be necessarily fruitful to relate such a generating function directly to physical observables.

5. Klein-Gordon Equation with a Potential

To extend the theory to include a potential, the case of a charged particle in an electrostatic field is considered. Assuming a time harmonic solution, Eq. (3), with an angular frequency $E/\hbar$, Eq. (7) becomes

$$c^2\hbar^2\nabla^2\psi/\psi + E^2 = m^2c^4. \tag{13}$$

The macroscopic energy equation for a charged particle in an electrostatic field is

$$-c^2p^2 + (E - e\phi)^2 = m^2c^4, \tag{14}$$

where ϕ is the electrostatic potential. Thus, one is led to postulate the time-independent relativistic wave equation[17]

$$c^2\hbar^2\nabla^2\psi + (E - e\phi)^2\psi = m^2c^4\psi. \tag{15}$$

For any potential function V, Eq. (15) may be generalized to yield

$$c^2\hbar^2\nabla^2\psi + [(E - V)^2 - m^2c^4]\psi = 0. \tag{16}$$

[17] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 306–309.

6. The Schrödinger Equation

To derive Schrödinger's nonrelativistic time-independent equation, the bracket appearing in Eq. (16) may be written

$$(E-V)^2-m^2c^4\approx 2mc^2(E'-V), \tag{17}$$

where E', the total nonrelativistic energy, and V are small compared with the rest mass energy. Substituting Eq. (17) into Eq. (16), dropping the prime, then gives the Schrödinger equation

$$\hbar^2\nabla^2\psi+2m(E-V)\psi=0. \tag{18}$$

This approximation does not imply that Eq. (18) is independent of relativistic effects, since de Broglie's[15] work indicates that quantum effects result, in part, as a consequence of relativity. This derivation of Schrödinger's equation makes it possible to assume pure real sine and cosine representations for the time harmonic solution instead of the complex representation required by traditional quantum theory.

7. Trajectories for a Nonrelativistic Particle

The angular frequency in the relativistic case as well as the nonrelativistic case, Eq. (18), has up to this point been given by the Planck-Einstein frequency condition using the total relativistic energy, Eq. (1). Separating the energy into the rest mass energy plus the total nonrelativistic energy, it is assumed that the very rapid fluctuations associated with the rest mass energy average out and produce no net observable effect in the present nonrelativistic limit. The time variation to be used in conjunction with Eq. (18) may, thus, be assumed to be given by

$$\sin(Et/\hbar) \quad \text{or} \quad \cos(Et/\hbar), \tag{19}$$

where E is now the nonrelativistic total energy.

Since the relativistic momentum reduces continuously to the classical momentum in the nonrelativistic limit, the first of Eqs. (10) may be used to obtain the momentum from the solution of the Schrödinger Eq. (18). From Eqs. (19) it is apparent that the second of Eqs. (10) may be used if the energy on the left is now interpreted as the classical total energy. Thus, formally the identical Eqs. (10) may be used to specify the trajectory of both a relativistic and a nonrelativistic particle.

Since the present total energy may be either positive or negative, whereas the relativistic total energy was always positive, an ambiguity in sign has been introduced in the nonrelativistic limit. The nonrelativistic total energy enters into Eqs. (19) merely as a magnitude to specify the unsigned period of the motion. It is found that for a free particle the sign in the second of Eqs. (14) need not be changed, but for a bound particle the trajectory should be specified by the prescription

$$\mathbf{p}=F'\nabla\psi, \quad |E|=F'\partial\psi/\partial t. \tag{20}$$

In the present nonrelativistic limit the momentum is given by the Newtonian expression

$$\mathbf{p}=m d\mathbf{r}/dt. \tag{21}$$

The trajectory of a nonrelativistic particle may be found by dividing Eqs. (10) [or Eqs. (20)] two at a time, using Eqs. (21), and integrating.

VI. PROBABILITY DISTRIBUTION

Experimentally, the initial conditions for submicroscopic particles are never known exactly, and many particles with many different initial conditions are observed. The raw data of an experiment then becomes a distribution function which is referred to as a "probability" distribution function. The classical probability per unit volume of finding a particle (or the center of mass of a small body) at a given point $\mathbf{r}'$ and at an instant t is given by the delta function

$$P_0=\delta[\mathbf{r}'-\mathbf{r}(\mathbf{r}_0,t)], \tag{22}$$

where $\mathbf{r}(\mathbf{r}_0,t)$ is the position of the particle as a function of time and the initial position $\mathbf{r}_0$.

Assuming an indefinitely large number of observations, the time average probability becomes

$$P_t=\frac{1}{t_2(\mathbf{r}_0)-t_1(\mathbf{r}_0)}\int_{t_1(\mathbf{r}_0)}^{t_2(\mathbf{r}_0)}\delta[\mathbf{r}'-\mathbf{r}(\mathbf{r}_0,t)]dt, \tag{23}$$

where the limits of integration are chosen so as to permit a proper time average.

Assuming that observations are made in ignorance of the initial position, it is then of interest to average over all possible initial positions. But since it may not always be possible to assume that all initial positions are equally accessible to the particle, as is usually the case in macroscopic mechanics, it is necessary to include a weighting function $f(\mathbf{r}_0)$. Instead of $\psi\psi^*$, the probability distribution function according to the present theory then becomes

$$P(\mathbf{r}')=\frac{1}{V_0}\int_{V_0}\frac{f(\mathbf{r}_0)}{t_2(\mathbf{r}_0)-t_1(\mathbf{r}_0)}d^3r_0 \times\int_{t_1(\mathbf{r}_0)}^{t_2(\mathbf{r}_0)}\delta[\mathbf{r}'-\mathbf{r}(\mathbf{r}_0,t)]dt, \tag{24}$$

where the weighting function $f(\mathbf{r}_0)$ may remain unknown for some problems and where V_0 is the volume of all possible initial values. Since this integral function, Eq. (24), depends upon the trajectory of the particle, $\mathbf{r}(\mathbf{r}_0,t)$, it can be evaluated only after the solution to a particular problem has been obtained. There are no requirements (as is the case for $\psi\psi^*$) that this function be regular; it may be infinite, discontinuous, and multiple valued. It is assumed, however, that it must always be integrable over all $\mathbf{r}'$ space to unity.

Because of the many successful conclusions which have been obtained by assuming ψ to be a probability amplitude, it is clear that Eq. (24) should be quite similar to $\psi\psi^*$. In principle, it should be possible to determine the difference between $\psi\psi^*$ and Eq. (24) experimentally.

VII. FREE PARTICLE

The solution to Schrödinger's nonrelativistic Eq. (18) [appropriately combined with Eqs. (19)] for a free particle traveling in the positive x direction is

$$\psi=\sin[(p_0x-Et)/\hbar], \tag{25}$$

where an additive phase constant has been neglected and where

$$p_0=(2mE)^{\frac{1}{2}}=mv_0. \tag{26}$$

Substituting Eq. (25) into Eqs. (10) then gives the momentum equations,

$$\begin{aligned} p&=F'(p_0/\hbar)\cos[(p_0x-Et)/\hbar],\\ E&=F'(E/\hbar)\cos[(p_0x-Et)/\hbar]. \end{aligned} \tag{27}$$

Dividing Eqs. (27) and using Eq. (21), the differential equation of motion becomes

$$p=mdx/dt=mv_0. \tag{28}$$

Integrating Eq. (28) then yields the expected trajectory

$$x=v_0t+x_0. \tag{29}$$

It is of interest to note that Eqs. (25), (27), and (29) are also valid in the relativistic case if E is interpreted as the total relativistic energy and ψ is regarded as a solution to the Klein-Gordon Eq. (7). Here, instead of Eq. (28) the velocity in terms of the energy, Eqs. (5), becomes

$$v_0=c(1-m^2c^4/E^2)^{\frac{1}{2}}. \tag{30}$$

It is also possible to obtain the free-particle trajectory from a bound particle as boundaries recede to infinity or potentials go to zero. Since a particle that is actually observed is a particle that has been captured or otherwise seriously interfered with, it is perhaps more realistic to assume that bound-particle problems have the greater physical significance.

VIII. PARTICLE IN A BOX

Because of the confinement between the walls of the box, the motion of the particle is repeated every half cycle instead of every full cycle; and the time variation is taken as

$$\sin(2Et/\hbar). \tag{31}$$

1. Position of the Particle as a Function of Time

The space part of ψ is given by the solution[18] of Schrödinger's Eq. (18) which vanishes at $x=0$ and L; thus,

$$\sin(n\pi x/L),\quad n=1,2,3,\cdots, \tag{32}$$

where the normalization factor has been neglected. The energy eigenvalues are

$$E=\pi^2\hbar^2n^2/2mL^2. \tag{33}$$

Multiplying Eqs. (31) and (32), substituting into Eqs. (20), where E has been replaced by $2E$, and using Eq. (21) yields

$$\begin{aligned} mdx/dt&=F'(n\pi/L)\sin(2Et/\hbar)\cos(n\pi x/L),\\ 2E&=F'(2E/\hbar)\cos(2Et/\hbar)\sin(n\pi x/L). \end{aligned} \tag{34}$$

Dividing Eqs. (34) and integrating gives the desired equation of motion,

$$\cos(n\pi x/L)=\cos(n\pi x_0/L)\cos(2Et/\hbar). \tag{35}$$

2. Fictitious Potential[19]

To analyze the motion, it is convenient to introduce the fictitious potential U defined by

$$U=E-m\dot{x}^2/2, \tag{36}$$

where the dot indicates differentiation with respect to time. Differentiating Eq. (35), the velocity may be expressed as a function of position,[20]

$$\dot{x}=\frac{n\pi\hbar}{mL}\left[1-\frac{\sin^2(n\pi x_0/L)}{\sin^2(n\pi x/L)}\right]^{\frac{1}{2}}. \tag{37}$$

Squaring Eq. (37) and substituting into Eq. (36), the fictitious potential becomes

$$U=E\sin^2(n\pi x_0/L)/\sin^2(n\pi x/L). \tag{38}$$

The oscillatory motion is now analyzed in terms of the particle moving in the presence of the potential given by Eq. (38). It is clear that the fictitious potential U is not a potential in the ordinary sense, since it depends upon the initial position of the particle x_0 as well as the boundaries of the box. A few of these potential curves are shown in Fig. 1. Because of the choice of the phase in the time function, Eq. (31), the initial velocity is zero. This means that the initial position x_0 has been chosen as a turning point of the motion. In one-dimensional motion only two initial conditions are needed; and since the total energy has already been specified as an eigenvalue, only the initial position or the initial velocity need be specified in addition.

3. Cellular Motion

By examining Eq. (38) or Fig. 1, it is seen that (except for $x_0=sL/n$, where $s=0, 1, 2,\cdots, n$) the mo-

[18] L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), p. 97.

[19] This simple device from classical mechanics, which is used to illustrate a solution already obtained, should not be confused with Bohm's quantum potential, $U=(-\hbar^2/2m)(\nabla^2R/R)$ where $\psi=R\exp(iS/\hbar)$.

[20] The present dynamical solution may be contrasted with the static solution of de Broglie and Bohm where the particle in a box remains rigidly fixed for all time.

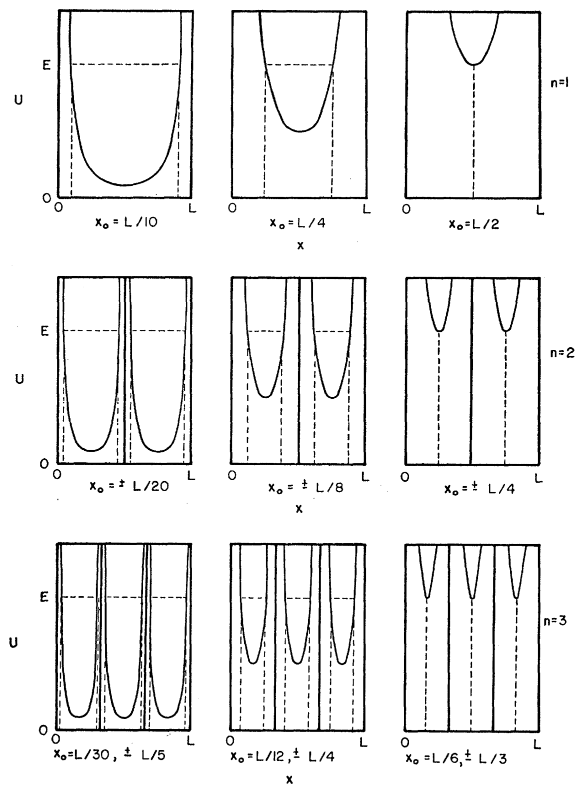

FIG. 1. Fictitious potential for a particle in a box, Eq. (38), for various initial positions x_0 and $n=1$, 2, and 3. The curves on the right show stagnation.

tion is confined to a cell of width $\Delta x = L/n$, the number of such cells in the box being n. The velocity of the particle as it moves back and forth in the cell may be pictured as the velocity that a particle would have sliding down the potential curves shown in Fig. 1. The turning points of the motion are given for zero velocity, $U=E$, or $x=x_0$. There are two conjugate turning points for each cell,

$$x_0 = x_1 + sL/n, \quad x_0' = -x_1 + (s+1)L/n, \tag{39}$$

where s is the number of the cell, $s=0, 1, 2, \cdots, n$, and where $0 \leq x_1 \leq L/2n$. The maximum velocity occurs at the minimum potential energy which according to Eq. (38) is at the center of the cell. This cellular type motion appears to be characteristic for bound particles in general.

4. Stagnation Points

If the initial position is chosen in the center of a cell, $x_0 = (s+\frac{1}{2})L/n$, it is found that the velocity is zero for all time. The center of the cells are stagnation points; once a particle is placed at a stagnation point with zero velocity it remains there without motion—even though the total energy is not zero but is given by Eq. (33). Such stagnation points appear to be characteristic of all bound particle problems.

5. Correspondence Principle

Examining the transition to macroscopic motion, it is apparent that a particle should not be confined to a cell. To find the conditions for which a particle need not be confined to a cell, we examine the case of x_0 approaching the boundary of a cell, $x_0 = \epsilon + sL/n$ for $\epsilon \rightarrow 0$. As long as ϵ is finite, the fictitious potential, Eq. (38), is infinite at $x = sL/n$, but in the limit as $\epsilon \rightarrow 0$, $U=0$. Since both an infinite potential and a zero potential may be interpreted as having physical significance here, a physical potential may be conceived of which approaches the mathematical expression Eq. (38) as a limit. Such a potential is shown qualitatively in Fig. 2. According to this picture the actual potential energy becomes zero for the entire motion if the initial position is taken on the cell boundary and a particle need not be confined to a cell but may move across the cells. Thus, for the initial position taken at the boundary of a cell, classical macroscopic motion is obtained. The particle moves with uniform velocity from one side of the box to the other.

For large energy E and, consequently, for large quantum numbers, the number of cells becomes large and the apparent macroscopic continuity of the region in the box is established. It may be verified that the period in Eq. (31) has been correctly chosen, since the time to traverse a cell in one direction is one-half the period; or,

$$(L/n)/\dot{x} = (2\pi\hbar/2E)/2, \tag{40}$$

which is seen to be satisfied upon substituting E as given by Eq. (33) and $\dot{x}$ as given by Eq. (37) for $x_0 = sL/n$ into Eq. (40).

Formally, the correspondence principle may be established by substituting $x_0 = sL/n$ into Eq. (35) to obtain

$$\cos(n\pi x/L) = \cos[(n\pi x_0/L) \pm (2Et/\hbar)], \tag{41}$$

or

$$x = x_0 \pm n\pi\hbar t/mL, \tag{42}$$

where Eq. (33) has been used.

6. Probability Distribution

For the one-dimensional motion of interest here, the probability distribution function, Eq. (24), becomes

$$P(x') = (2E/\pi\hbar L)\int_0^L dx_0\, f(x_0)\int_0^{\pi\hbar/2E} dt \times \delta[x' - x(x_0,t)], \tag{43}$$

where the particle is assumed to be confined to one cell, so that the time average is taken over half the period, the total period being $\pi\hbar/E$. Changing the integration with respect to the time t to an integration with respect to x and integrating, dropping primes, we obtain

$$P(x)=(2E/\pi\hbar L)\int_0^L dx_0 \frac{f(x_0)}{|\dot{x}(x,x_0)|} \times\{u[x-x_0]-u[x+x_0-(2s+1)L/n]\}, \quad (44)$$

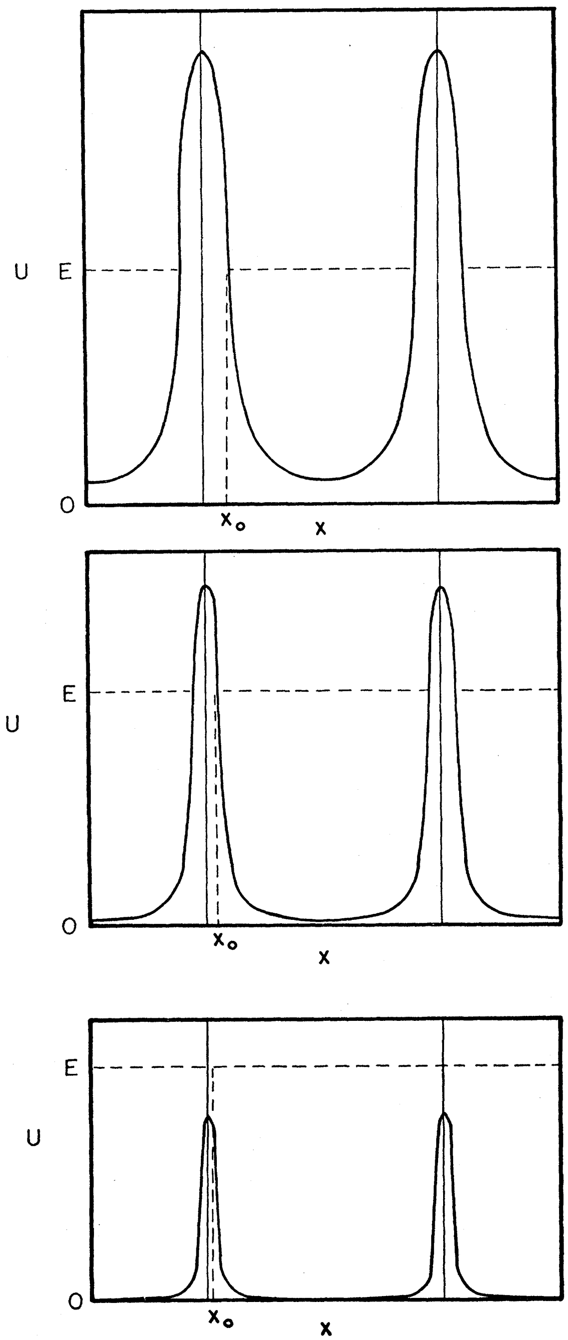

FIG. 2. An exaggerated diagram showing how the fictitious potential U is assumed to vanish as the initial position x_0 approaches a cell boundary.

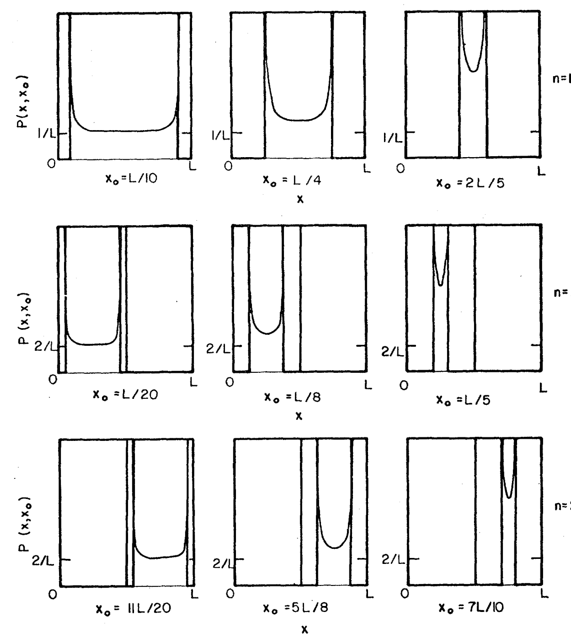

FIG. 3. The individual-particle probability for a particle in a box, Eq. (46), for various initial positions x_0 and $n=1$ and 2.

where $s=0, 1, 2,\cdots, (n-1)$ is the cell in which the motion occurs and where $u(x-x_0)$ is the unit step function which is 0 for $x<x_0$ and 1 for $x\geq x_0$. Substituting the value of $\dot{x}$ as given by Eq. (37) into Eq. (44) and limiting the integration to the cell in which the motion occurs, it is found that

$$P(x)=(2n/L^2)|\sin(n\pi x/L)|\int_{sL/n}^{x} dx_0\, f(x_0) \times[\sin^2(n\pi x/L)-\sin^2(n\pi x_0/L)]^{-\frac{1}{2}}, \quad (45)$$

where the factor 2 has been included to take care of the equivalent contribution from $x_0=-x+(2s+1)L/n$ to $x_0=(s+1)L/n$.

The probability distribution for a single particle for a particular choice of the initial position x_0 may be obtained by omitting the averaging integration over x_0; thus from Eqs. (44) and (37),

$$P(x,x_0)=\frac{n}{L}|\sin(n\pi x/L)| \times\frac{u[x-x_0]-u[x+x_0-(2s+1)L/n]}{[\sin^2(n\pi x/L)-\sin^2(n\pi x_0/L)]^{\frac{1}{2}}}. \quad (46)$$

This function, Eq. (46), is shown in Fig. 3.

To evaluate the integral in Eq. (45), it is necessary to specify the function $f(x_0)$. The only restriction on the nature of $f(x_0)$ appears to be an integrability requirement which may be obtained by integrating $P(x)$ from $x=0$ to L and setting the result equal to unity. Thus, from Eqs. (44) and (37) it is possible to obtain

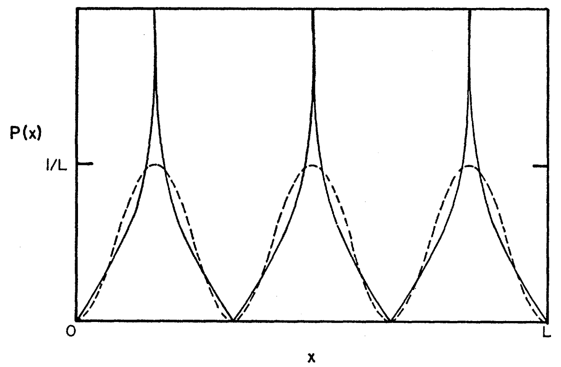

FIG. 4. The many-particle probability distribution for a particle in a box, Eq. (49), for $n=3$. The dashed line gives $\psi\psi^*$, Eq. (50), for comparison.

the expected condition

$$(1/L)\int_0^L dx_0\, f(x_0)=1. \qquad (47)$$

Macroscopically, it can be assumed that all initial positions, x_0, are equally likely, or $f(x_0)=1$. With this macroscopic assumption, it is of interest to compute the probability distribution that results. Setting $f(x_0)=1$ in Eq. (45) and changing the variable of integration from x_0 to w where

$$\sin(n\pi x_0/L)=\sin(n\pi x/L)\sin w, \qquad (48)$$

the integral is seen to be the complete elliptic integral of the first kind[21]; which gives the result

$$P(x)=(2/\pi L)\,|\sin(n\pi x/L)|\,K[\sin(n\pi x/L)]. \quad (49)$$

This probability distribution function is shown in Fig. 4 for $n=3$ together with the traditional quantum mechanical expression,

$$\psi\psi^*=(2/L)\sin^2(n\pi x/L), \qquad (50)$$

where ψ has been obtained from Eq. (32) by including an appropriate normalizing factor. Despite the functional and numerical differences between $P(x)$, Eq. (49), and $\psi\psi^*$, Eq. (50), it is interesting to note some of the similarities. Both the present theory and the traditional quantum theory yield the same number of cells, the same points of zero probability, the same points of maximum probability, and, of course, the same area under the curves. Contrary to traditional quantum theory the present theory, with $f(x_0)$ chosen as unity, yields infinity at the points of maximum probability.

IX. DOUBLE SLIT

1. Wave Function ψ

It is extremely difficult to obtain a solution of Schrödinger's Eq. (18) adequate for all ranges of the parameters for the double slit problem. Some of the physics may, however, be investigated by considering the asymptotic solution for a plane wave incident normally upon a screen with two infinitesimally narrow slits separated by a distance D. Because of the cylindrical geometry, the asymptotic solution is taken to be

$$\psi\sim R_1^{-\frac{1}{2}}\sin[(p_0R_1-Et)/\hbar] + R_2^{-\frac{1}{2}}\sin[(p_0R_2-Et)/\hbar], \quad (51)$$

where p_0 is given by Eq. (26) and R_1 and R_2 are the distances from the two slits to the point in question. Again, only a pure real representation has been chosen. A normalization factor has been omitted in Eq. (51).

Introducing elliptic cylinder coordinates,

$$\xi=(R_1+R_2)/D,\quad \eta=(R_1-R_2)/D, \qquad (52)$$

the asymptotic wave function, Eq. (51), becomes

$$\psi\sim(1/\xi^{\frac{1}{2}})\sin(k\xi-\omega t)\cos k\eta, \qquad (53)$$

where

$$k=p_0D/2\hbar=(2mE)^{\frac{1}{2}}D/2\hbar,\quad \omega=E/\hbar. \qquad (54)$$

To satisfy the boundary condition that $\psi=0$ for $\eta=\pm1$, the quantum condition

$$k=(2n+1)\pi/2 \quad\text{or}\quad E=(2n+1)^2\hbar^2\pi^2/2mD^2, \quad (55)$$

must be satisfied. This condition is the same as for the even modes (n odd) of a particle in a box of width D, Eq. (33). A double slit thus permits only particles with certain discrete energies, Eq. (55), to pass through. Conditions other than Eq. (55) would apply for finite width slits and other angles of incidence.

2. Particle Trajectories

In elliptic cylinder coordinates, the momentum according to Eqs. (10) may be written

$$\begin{aligned} mDh_1d\xi/dt&=F'(2/Dh_1)\partial\psi/\partial\xi,\\ mDh_2d\eta/dt&=F'(2/Dh_2)\partial\psi/\partial\eta,\\ E&=-F'\partial\psi/\partial t,\end{aligned} \qquad (56)$$

where

$$h_1^2=(\xi^2-\eta^2)/(\xi^2-1),\quad h_2^2=(\xi^2-\eta^2)/(1-\eta^2). \quad (57)$$

Substituting Eq. (53) into Eqs. (56), noting that for the present asymptotic limit $\xi^2\gg1\geq\eta^2$, and dividing two at a time, the differential equations of motion for the particle become

$$\begin{aligned} kd\xi&=\omega dt,\\ \xi^{-2}\tan(k\xi-\omega t)d\xi&=-(1-\eta^2)^{-1}\cot k\eta d\eta.\end{aligned} \qquad (58)$$

Integrating the first of Eqs. (58), the motion is found to be uniform in the ξ direction,

$$k\xi=\omega t+k\xi_0, \qquad (59)$$

where $\xi_0\gg1$ is the initial position. Substituting Eq. (59) into the second of Eqs. (58) and considering points near the plane of symmetry, $\eta^2\ll1$, the resulting ex-

[21] P. Franklin, *Methods of Advanced Calculus* (McGraw-Hill Book Company, Inc., New York, 1944), p. 274.

pression may be integrated to yield

$$\sin k\eta = \sin k\eta_1 \exp[(k \tan k\xi_0)/\xi], \tag{60}$$

where

$$\eta = \eta_1, \tag{61}$$

describes the hyperbolic trajectory for $\xi \to \infty$.

A further insight into the motion may be obtained by considering the time rate of change of η as given by Eqs. (56), (59), and (60),

$$d\eta/dt = -(\omega/k^3) \cot k\xi_0 \tan k\eta \ln^2(\sin k\eta/\sin k\eta_1). \tag{62}$$

Stagnation trajectories, defined as the trajectories along which $d\eta/dt=0$, are given by the hyperbolas

$$\eta = n\pi/k, \quad n=0, \pm1, \pm2, \cdots \tag{63}$$

These trajectories coincide with the observed interference maxima. Cell boundaries, defined as the curves along which $d\eta/dt \to \pm\infty$, are given by the hyperbolas

$$\eta = (2n+1)\pi/2k, \quad n=0, \pm1, \pm2, \cdots \tag{64}$$

These hyperbolic cylindrical surfaces are the nodal surfaces of the interference pattern. Since the particle moves from these surfaces with infinite velocity, no particles will be found on the nodal surfaces. The motion is confined to the region between the nodal surfaces (see Fig. 5).

The variation of η with time, as may be deduced from Eqs. (59), (60), and (62), is not oscillatory. From the initial position η_0, η varies monotonically with time to the asymptote η_1. The initial and final positions are such that the particle moves away from the nodal surfaces, Eq. (64), and toward surfaces of maxima, Eq. (63).

Since the asymptotic solution is not valid in the neighborhood of the slits, it is of some interest to indicate some qualitative features of the solution near the slits. Since the boundary conditions requires $\psi=0$ for $\xi=1$ (as well as for $\eta=\pm1$), the surfaces of maxima (and, therefore, the nodal surfaces as well) terminate at one slit or the other and not in the region between the slits as might be erroneously concluded from the asymptotic solution alone, Eq. (63). From symmetry and from the fact that a trajectory never crosses a nodal surface, particles that pass through the upper slit (see Fig. 5) where $\eta=1$ will always remain in the upper half-space $0\leq\eta\leq1$; and particles that pass through the lower slit where $\eta=-1$ will always remain in the lower half-space $-1\leq\eta\leq0$.

To obtain an estimate of the particle probability distribution function corresponding to the interference pattern requires a knowledge of initial conditions and, therefore, a more complete solution of Schrödinger's Eq. (18) for all ranges of the parameters—an undertaking beyond the scope of this paper.

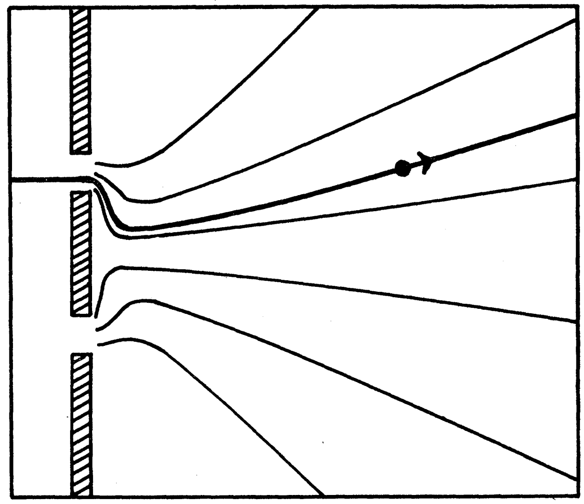

FIG. 5. A sketch to indicate how the trajectory of a particle passing through a double slit is confined between nodal surfaces.

X. CONCLUSIONS

Although the present theory appears to resolve a number of difficulties present in the traditional quantum theory and suggests that submicroscopic phenomena may be amenable to investigation with the fruitful tools of classical physics, it still fails to present the actual classical problem being solved. Although a simple prescription for finding more-or-less reasonable trajectories, which seem to be superior to those prescribed by previous causal theories, has been given, the classical situation that gives rise to this prescription remains obscure.

Since the wavelike appearance of particles is assumed here to be only approximate, eventually it should be possible to suggest an ordinary experiment that will distinguish between the theory presented here and other theories. No experiment is suggested at this time because of the difficulty, largely mathematical, in ascertaining $f(\mathbf{r}_0)$, Eq. (24).

The theory presented here yields exactly the same energy eigenvalues, energy differences, and wave functions as the traditional quantum theory; therefore, it may be assumed that perturbation theory, transition probabilities, and quantum statistics remain unchanged. The so-called penetration of a potential barrier by a particle, although requiring a different physical interpretation, can be expected to be formally similar to the traditional quantum theory. The present theory, in common with all causal theories, requires an explanation for the apparently random emission of radioactive particles—contrary to the traditional theory which assumes the radioactive decay process to be intrinsically chaotic.

Further work is needed to extend the present theory beyond the case of a single scalar particle.

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