Displacement current

— and how to get rid of it

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To enable the continuity of electric current to be retained across a capacitor Maxwell proposed a "displacement current". By treating the capacitor as a special kind of transmission line this mathematical convenience is no longer required.

CONVENTIONAL electromagnetic theory proposes that when an electric current flows down a wire into a capacitor it spreads out across the plate, producing an electric charge which in turn leads to an electric field between the capacitor plates. The valuable concept of continuity of electric current is then retained by postulating (after Maxwell)[1] a "displacement current", which is a mathematical manipulation of the electric field E between the capacitor plates which has the dimensions of electric current and completes the flow of "electricity" (Fig. 1 (a) and (b)). This approach permits us to retain Kirchhoff's Laws and other valuable concepts, even though superficially it appears that at the capacitor there is a break in the otherwise continuous flow of electric current.

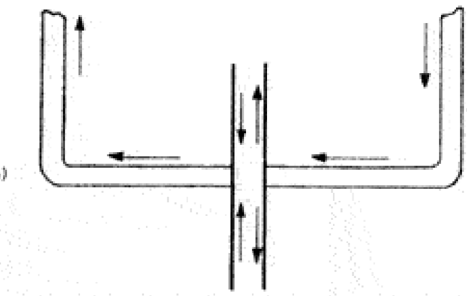
The flaw in this model is revealed when we notice that the electric current entered the capacitor at one point only on the capacitor plate. We must then explain how the electric charge flowing down the wire suddenly distributes itself uniformly across the whole capacitor plate. We know that this cannot happen since charge cannot flow out across the plate at a velocity in excess of the velocity of light. This paradoxical situation is brought about by a fundamental flaw in the basic model. Work on high speed logic design[2] has shown that the model of a lumped capacitance is faulty, and "displacement current" is an artefact of this faulty model.

The true model is quite different. Electric current enters the capacitor through a wire and then spreads out across the plate of the capacitor in the same way as ripples flow out from a stone dropped into a pond. If we consider only one pie-shaped wedge of the capacitor, as in Fig 1 (c), we can recognise it as a parallel plate transmission line whose only unusual feature is that the line width is increasing (and hence the impedance is decreasing). The capacitor is made up of a number of these pie-shaped transmission lines in parallel, so the proper model for a capacitor is a transmission line.

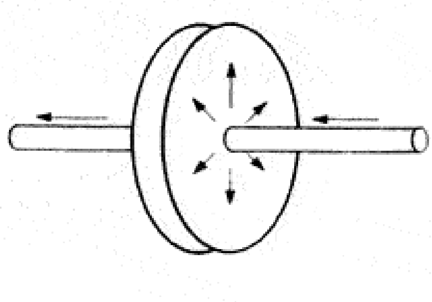
Equivalent series resistance for a capacitor is the initial characteristic impedance of this transmission line at a radius equal to the radius of the input wires. Series inductance does not exist. *Pace* the many documented values for series inductance in a capacitor, this confirms experience that when the so-called series inductance of a capacitor is measured it turns out to be no more than the series inductance of the wires connected to the capacitor. No mechanism has ever been proposed for an internal series inductance in a capacitor.

Since any capacitor has now become a transmission line, it is no more necessary to postulate "displacement current" in a capacitor than it is necessary to do so for a transmission line. The excision of "displacement current" from Electromagnetic Theory has been based on arguments which are independent of the classic dispute over whether the electric current causes the electromagnetic field or vice versa.

(a)



(b)



(c)

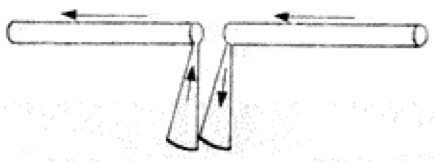


Fig. 1 *Process of current flowing into a capacitor and spreading out across a plate is shown in (a) and (b). The structure in (b) can be considered as being made up of a number of pie-shaped wedges as in (c), each of which is a transmission line.*

Appendix

Comparison of the transmission line model with the lumped model of a capacitor in an RC circuit.

Taking the above discussion further, consider a transmission line as shown in Fig. 2, assumed to be terminated with a resistance R_T (not shown). The reflection coefficient is $\rho=(R_T-Z_o)/(R_T+Z_o)$ where Z_o is the characteristic impedance of the line. If the line is open-circuit at the right-hand end, as shown (and therefore R_T is infinite), the $\rho=+1$. We will assume that $R \gg Z_o$.

When switch S is closed (at time $t=0$) a step of voltage $V.Z_0/(R+Z_o)$ is propagated down the line. This reflects from the open circuit at the right hand end to give a total voltage $2V.Z_o/(R+Z_o)$. Reflection from the left end makes a further contribution of $[V.Z_0/(R+Z_0)] \times [(R-Z_0)/(R+Z_0)]$ and so on. In general after n two-way passes the voltage after n passes is V_n and,

$$V_{n+1}=V_n+2.\frac{VZ_0}{R+Z_0}\left[\frac{R-Z_0}{R+Z_0}\right]^n \quad (1)$$

In order to avoid a rather difficult integration it is possible to sum this series to n terms using the formula,

$$=\frac{a(1-v^n)}{1-v} \quad (2)$$

where a is the first term of a geometrical progression and v the ratio between terms. (This formula is easily verified by induction.) Substituting in (2) the parameters from (1),

$$\text{i.e. } a=\frac{2VZ_0}{R+Z_0} \quad (3)$$

$$v=\frac{R-Z_0}{R+Z_0} \quad (4)$$

We obtain,

$$V_n = \frac{\dfrac{2VZ_0}{R+Z_0}\left[1-\left[\dfrac{R-Z_0}{R+Z_0}\right]^n\right]}{1-\dfrac{R-Z_0}{R+Z_0}} \qquad (5)$$

$$= V\left[1-\left[\frac{R-Z_0}{R+Z_0}\right]^n\right] \qquad (6)$$

This is the correct description of what is happening as a capacitor charges. We can now go on to show that it is approximated by an exponential. We have

$$V_n = V\left[1-\left[\frac{R-Z_0}{R+Z_0}\right]^n\right] \qquad (7)$$

Consider the term,

$$T = \left(\frac{R-Z_0}{R+Z_0}\right)^n$$

$$= \left(\frac{1-Z_0/R}{1+Z_0/R}\right)^n$$

If $Z_0/R \ll 1$ this term is asymptotically equal to

$$\left(1-\frac{2Z_0}{R}\right)^n$$

Now define $k = 2Z_0 n/R$. Substitution gives:

$$T = \left[1-\frac{k}{n}\right]^n$$

By definition, as $n \to \infty$ we have,

$$T = e^{-k} = e^{\frac{-2Z_0 n}{R}}$$

And therefore:

$$V_n = V\left[1-e^{\frac{-2Z_0 n}{R}}\right]$$

Now, after time t, $n = V_c t/2l$, where V_c = velocity of propagation.

Therefore

$$V(t) = V\left[1-e^{\frac{-V_c t}{l}\cdot\frac{Z_0}{R}}\right]$$

For any transmission line it can be shown that:

$$Z_0 = f\sqrt{\frac{\mu}{\epsilon}}$$

$$V_c = \frac{1}{\sqrt{\mu\epsilon}}$$

$$C_1 = \epsilon/f$$

where C_1 = capacitance per unit length, and f is the same geometrical factor in each case. The "total capacitance" of length l of line $= l.C_1 = C$.

$$\text{Hence } \frac{V_c Z_0}{lR} = \frac{1}{RC}$$

and therefore

$$V(t) = V(1-e^{-t/RC})$$

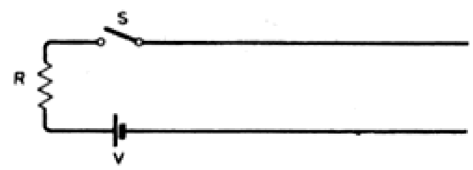
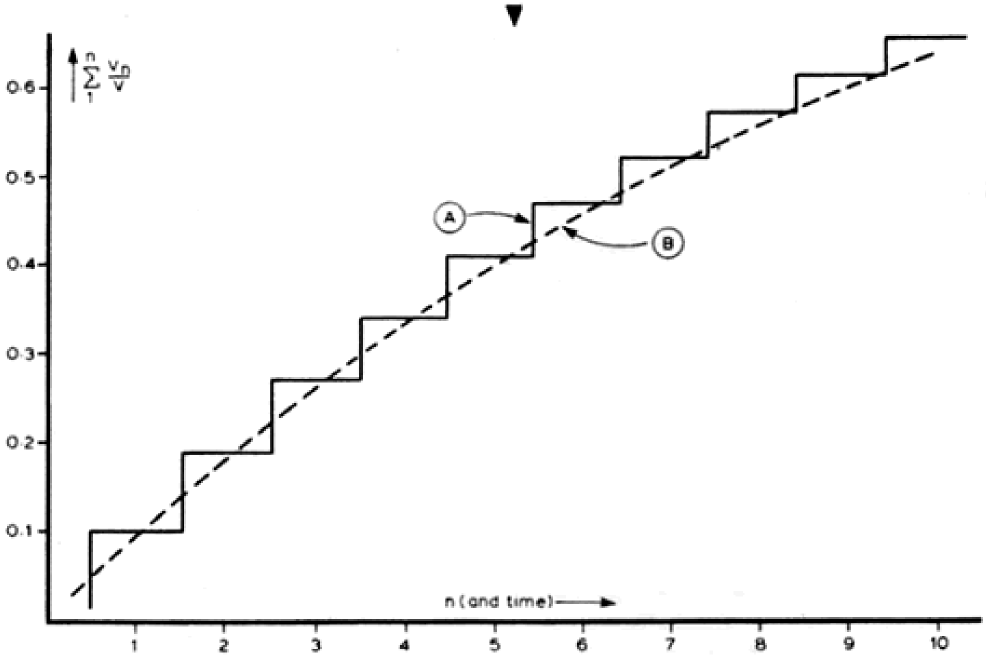


Fig. 2 *An open-ended transmission line.*

Fig. 3 *Comparison of the transmission line model $1-(1-2Z_0/R)^n$ in the curve A with the lumped model $1-e^{-2Z_0 n/R}$ in curve B, for $2Z_0/R = 0.1$.*



which is the standard result. This model does not require use of the concept of charge. A graphical comparison of the results is shown in Fig. 3.

References

1. "History of displacement current", I. Catt, M. F. Davidson, D. S. Walton. *Physics Education*, to be published early 1979.

2. "Crosstalk (noise) in digital computers", I. Catt. *IEEE Trans.* EC-16, Dec. 1967, pp. 743-763.