Battery and resistor. Steady state.

We start with a conventional view of a battery with voltage V connected via two uniform perfect conductors to a resistor R (Fig.1).

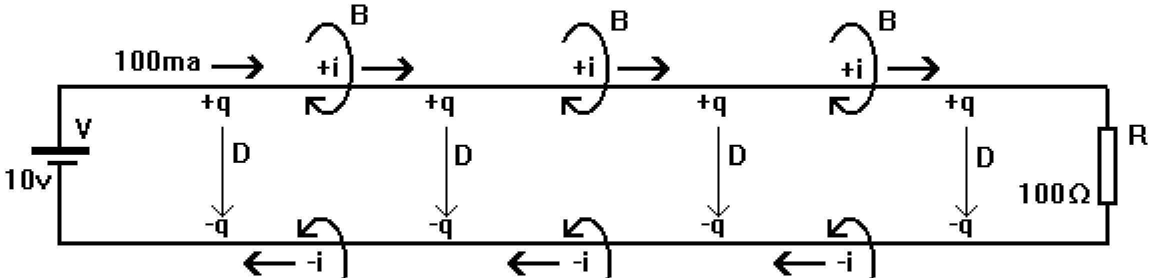

Figure 1

A steady current flows round the circuit, through battery, conductors and resistors. Ohm's Law tells us that the voltage equals the current multiplied by the resistance. Therefore the current is I = V/R. Every point on the surface of the upper conductor is at potential V, and every point on the surface of the lower conductor is at a zero potential.

The space between the two conductors, shown in cross section (Fig. 2), is filled by tubes of electric displacement D.

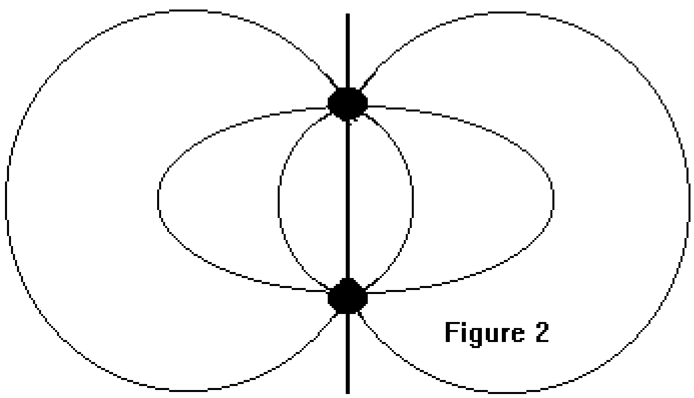
Figure 2

Each tube of electric displacement terminates on unit positive charge on the upper conductor and unit negative charge on the lower conductor [1]. If the capacitance between the two conductors is C, then the total charge on each conductor is given by Q= CV. If the capacitance per unit length is c, then the total charge per unit length on each conductor is q=cV

The energy stored in the electric field between the conductors is

$$U_d = \frac{1}{2} CV^2 \quad \text{(Ref. 1), or}$$

$$u_d = \frac{1}{2} cV^2 \quad \text{per unit length.}$$

The space between the two conductors is filled by tubes of magnetic flux which encircle the current in the conductor.

If the self inductance of the pair of conductors is L, then the total magnetic flux passing between the conductors is

$$\Phi = L\,I\,.$$

If the self inductance per unit length is *l*, then the magnetic flux per unit length is

$$\phi = l\,I$$

The energy stored in the magnetic field created by the current in the two conductors is

$$U_b = \frac{1}{2} L I^2 \text{, or}$$

$$u_b = \frac{1}{2} l I^2 \text{ per unit length (ref. 2)}$$

Power is delivered by the battery into the circuit at a rate of watts which is the product of voltage and current VI. The resistor absorbs power at the same rate, turning electric power into heat, which then radiates from it.

The energy trapped in the fields between the conductors totals

$$U_d + U_b = \frac{1}{2} CV^2 + \frac{1}{2} L I^2 \quad (1)$$

The energy stored in each unit length is

$$u_d + u_b = \frac{1}{2} cV^2 + \frac{1}{2} l I^2 \quad (2)$$

Battery and resistor. Initial state.

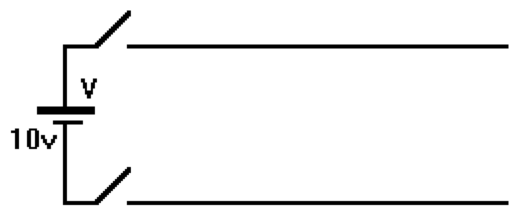

Figure 3

Now let us turn to the conventional view of the initial conditions. We will insert two switches, one in the top conductor and one in the bottom conductor (Fig.3). When we close the two switches, the distant resistor cannot define the current which rushes along the wires because the wave front has not yet reached the resistor (Figs.4,5).

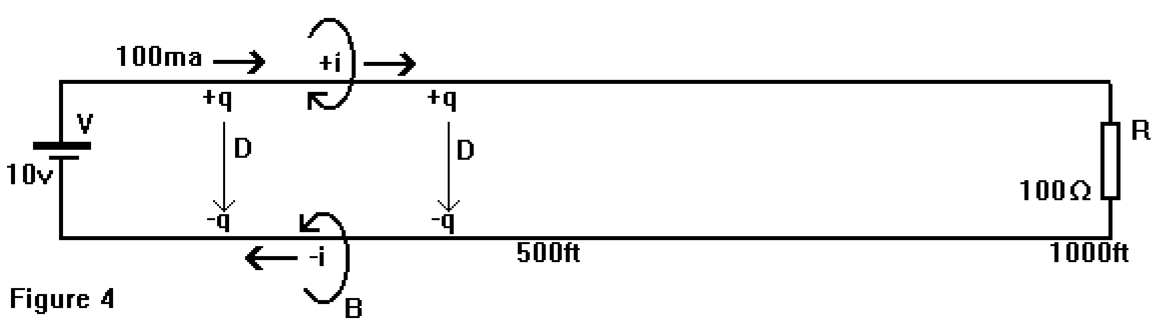

Figure 4

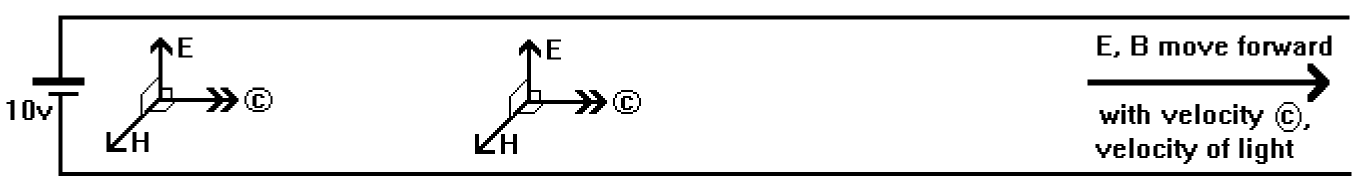

Figure 5

Lacking knowledge of the value of the resistor, the current is defined by the characteristic resistance Z_0 of the pair of conductors (usually called their characteristic impedance). Thus,

$$\frac{V}{I} = Z_0 .$$

So the instantaneous current is

$$\frac{V}{Z_0} .$$

Instead of delivering this power to the resistor, the battery delivers it into the space between the conductors for the first few nanoseconds. The wave front travels to the right at the speed of light for the vacuum ***C***. In our case, where the resistor is at a distance S from the battery, the wave front reaches the resistor after a time S/***C***. During this initial time, the battery supplies the energy necessary (eqn.1) to set up the electric and magnetic fields in the space between the conductors. The energy delivered by the battery during the time S/***C*** when the wave front travels from battery to resistor is VI S/***C***.

The characteristic resistance is

$$Z_0 = \sqrt{\frac{l}{c}} \quad \text{(Ref. 3)} \quad (3)$$

Simple algebra will show that in the initial (transient) case, electric and magnetic energy are equal (to u_1), as follows.

The energy in the electric field is

$$\mathbf{u_d} = \frac{1}{2} cV^2 \quad \text{(Ref. 1)}$$

Now

$$\frac{V}{I} = Z_0 \quad (\text{i.e. } V^2 = I^2 Z_0^{\ 2})$$

We can rewrite

$$\frac{1}{2} cV^2 \text{ as } \frac{1}{2} cI^2 Z_0^{\ 2}$$

Now substitute

$$\frac{l}{c} \text{ for } Z_0^{\ 2} \quad \text{(eqn.3)}$$

and we get

$$\frac{1}{2} l I^2 = (u_b),$$

the energy in the magnetic field.
Therefore

$$u_d = u_b = u_1$$

Now let us show that the energy (which we shall call $2u_2$) delivered by the battery in time 1/***C*** equals the energy stored in the fields ($2u_1$) in a section of unit length. Power from the battery is VI. One second's worth of this power charges up a length ***C***. So the energy stored in unit length is

$$2u_2 = \frac{VI}{C},$$

where ***C*** is the velocity of light. But we know that

$$C = \frac{1}{\sqrt{lc}} \text{ (Ref. 4)}$$

So VI/***C*** becomes

$$VI\sqrt{lc}$$

Substitute for I using the formula

$$I = \frac{V}{Z_0},$$

to give

$$2u_2 = \frac{V^2}{Z_0}\sqrt{lc}$$

Then using the formula (3) for Zo we end up with

$$2u_2 = cV^2,$$

which is twice the energy

$$u_d \ (= \frac{1}{2} cV^2)$$

in the electric field. Therefore

$$2u_1 = 2u_2 \ (=2u)$$

If the terminating resistor is equal to the transmission line's characteristic impedance, then there is no reflection. The battery thinks the transmission line has infinite length. It continues to deliver power at the initial rate.

Unterminated transmission line.

If the resistor is missing, then all of the energy travelling to the right at the speed of light is reflected and begins the return journey to the left.

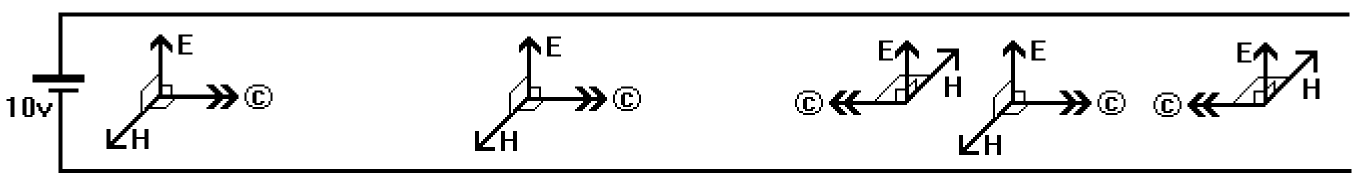

Figure 6

Let us consider the case where the line length is S, and time 3S/2***C*** has elapsed since the switches were closed (Fig.6). The field situation in the first half is as before, the energy per unit length being VI/***C***; half of it

$$u = \frac{1}{2}\frac{VI}{C}$$

in the electric field and half in the magnetic field. In the last half, a returning wave front of equal energy density is superposed on the energy making its outward journey. Magnetic fields cancel out, and the second half appears to be a steady charged capacitor, charged to an amplitude 2V. Our formula

$$U = \frac{1}{2}cV^2$$

is thought to give us the electric field's energy per unit length. Since the voltage has doubled, the energy appears to have quadrupled to

$$\frac{1}{2}\,c(2V)^2 = 2cV^2 = (4u)$$

instead of the

$$\frac{1}{2}\,cV^2 \quad (= u)$$

associated previously with the single electric field. Thus, double the electric field has led to four times the energy because the formula for energy contains the square of the voltage. This quadrupling is untrue, because the two electric fields, one travelling to the right and the other to the left, have no relationship with each other. The reality is that each electric field contains energy u per unit length, totalling 2u, not 4u, of electric energy. The missing energy is contained in the invisible magnetic fields. These are invisible because the leftwards travelling magnetic field makes the equal rightwards travelling magnetic field invisible to our measuring instruments. Thus, in the last half, the energy per unit length is made up as follows; u in the forward travelling electric field, u in the forward travelling magnetic field, u in the backward travelling electric field, and u in the backward travelling magnetic field. It is a mathematical accident that we get the correct answer for total energy when we wrongly think that the last half of the transmission line is steadily charged with electric field, and no magnetic field exists. Pace our calculations, the total energy density from electric fields is 2u not 4u.

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[1] This is Gauss's Law, which later became one of Maxwell's Equations.